

# RATIONALITY PROBLEM FOR ALGEBRAIC TORI

AKINARI HOSHI AND AIICHI YAMASAKI

**ABSTRACT.** We give a birational classification of algebraic tori of dimensions 4 and 5 over a field  $k$ . In particular, a birational classification of norm one tori whose Chevalley modules are of rank 4 and 5 is given. We show that there exist exactly 487 (resp. 7, resp. 216) stably rational (resp. not stably but retract rational, resp. not retract rational) algebraic tori of dimension 4, and there exist exactly 3051 (resp. 25, resp. 3003) stably rational (resp. not stably but retract rational, resp. not retract rational) algebraic tori of dimension 5. We make a procedure to compute a flabby resolution of a  $G$ -lattice effectively by using the computer algebra system GAP. Some algorithms may determine whether the flabby class of a  $G$ -lattice is invertible (resp. zero) or not. Using the algorithms, we determine all the flabby and coflabby  $G$ -lattices of rank up to 6. Moreover, we show that they are stably permutation. We also verify that the Krull-Schmidt theorem for  $G$ -lattices holds when the rank  $\leq 4$ , and fails when the rank is 5. Indeed, there exist exactly 11 (resp. 131)  $G$ -lattices of rank 5 (resp. 6) which are decomposable into two different ranks. Moreover, when the rank is 6, there exist exactly 18  $G$ -lattices which are decomposable into the same ranks but the direct summands are not isomorphic. As an application of the methods developed, some examples of not retract (stably) rational fields over  $k$  are given.

## CONTENTS

1. Introduction	2
2. Preliminaries: Tate cohomology and flabby resolutions	14
3. CARAT code of the $\mathbb{Z}$ -classes in dimensions 5 and 6	17
4. Krull-Schmidt theorem fails for dimension 5	18
4.0. Classification of indecomposable maximal finite groups of dimension $n \leq 6$	21
4.1. Krull-Schmidt theorem (1)	24
4.2. Krull-Schmidt theorem (2)	31
5. GAP algorithms: the flabby class $[M_G]^{fl}$	38
5.0. Determination whether $M_G$ is flabby (coflabby)	39
5.1. Construction of the flabby class $[M_G]^{fl}$ of the $G$ -lattice $M_G$	40
5.2. Determination whether $[M_G]^{fl}$ is invertible	43
5.3. Computation of $E$ with $[[M_G]^{fl}]^{fl} = [E]$	45
5.4. Possibility for $[M_G]^{fl} = 0$	47
5.5. Verification of $[M_G]^{fl} = 0$ : Method I	50
5.6. Verification of $[M_G]^{fl} = 0$ : Method II	54
5.7. Verification of $[M_G]^{fl} = 0$ : Method III	60
6. Flabby and coflabby $G$ -lattices	66
7. Norm one tori	77
8. Tate cohomology: GAP computations	80
9. Proof of Theorem 1.25	84
10. Proof of Theorem 1.26	96
11. Proof of Theorem 11.3	105
12. Application of Theorem 11.3	110
13. Tables for a birational classification of the algebraic $k$ -tori of dimension 5	113
References	124

---

2010 *Mathematics Subject Classification.* Primary 11E72, 12F20, 13A50, 14E08, 20C10, 20G15.

*Key words and phrases.* Rationality problem, birational classification, algebraic tori, stably rational, retract rational, flabby resolution, Krull-Schmidt theorem.

This work was partially supported by KAKENHI (22740028, 24540019). Some part of this work was done during the authors visited National Taiwan University (Department of Mathematics), whose support is gratefully acknowledged.

## 1. INTRODUCTION

Let  $k$  be a field and  $K$  be a finitely generated field extension of  $k$ . A field  $K$  is called *rational over  $k$*  (or  *$k$ -rational* for short) if  $K$  is purely transcendental over  $k$ , i.e.  $K$  is isomorphic to  $k(x_1, \dots, x_n)$ , the rational function field over  $k$  with  $n$  variables  $x_1, \dots, x_n$  for some integer  $n$ .  $K$  is called *stably  $k$ -rational* if  $K(y_1, \dots, y_m)$  is  $k$ -rational for some algebraically independent elements  $y_1, \dots, y_m$  over  $K$ .  $K$  is called *retract  $k$ -rational* if there is a  $k$ -algebra  $R$  contained in  $K$  such that (i)  $K$  is the quotient field of  $R$ , and (ii) the identity map  $1_R : R \rightarrow R$  factors through a localized polynomial ring over  $k$ , i.e. there is an element  $f \in k[x_1, \dots, x_n]$ , which is the polynomial ring over  $k$ , and there are  $k$ -algebra homomorphisms  $\varphi : R \rightarrow k[x_1, \dots, x_n][1/f]$  and  $\psi : k[x_1, \dots, x_n][1/f] \rightarrow R$  satisfying  $\psi \circ \varphi = 1_R$  (cf. [Sal84a]).  $K$  is called  *$k$ -unirational* if  $k \subset K \subset k(x_1, \dots, x_n)$  for some integer  $n$ . It is not difficult to see that “ $k$ -rational”  $\Rightarrow$  “stably  $k$ -rational”  $\Rightarrow$  “retract  $k$ -rational”  $\Rightarrow$  “ $k$ -unirational”.

Let  $L$  be a finite Galois extension of  $k$  and  $G = \text{Gal}(L/k)$  be the Galois group of the extension  $L/k$ . Let  $M = \bigoplus_{1 \leq i \leq n} \mathbb{Z} \cdot u_i$  be a  $G$ -lattice with a  $\mathbb{Z}$ -basis  $\{u_1, \dots, u_n\}$ , i.e. finitely generated  $\mathbb{Z}[G]$ -module which is  $\mathbb{Z}$ -free as an abelian group. Let  $G$  act on the rational function field  $L(x_1, \dots, x_n)$  over  $L$  with  $n$  variables  $x_1, \dots, x_n$  by

$$(1) \quad \sigma(x_i) = \prod_{j=1}^n x_j^{a_{i,j}}, \quad 1 \leq i \leq n$$

for any  $\sigma \in G$ , when  $\sigma(u_i) = \sum_{j=1}^n a_{i,j} u_j$ ,  $a_{i,j} \in \mathbb{Z}$ . The field  $L(x_1, \dots, x_n)$  with this action of  $G$  will be denoted by  $L(M)$ . There is the duality between the category of  $G$ -lattices and the category of algebraic  $k$ -tori which split over  $L$  (see [Vos98, page 27, Example 6]). In fact, if  $T$  is an algebraic  $k$ -torus, then the character group  $X(T) = \text{Hom}(T, \mathbb{G}_m)$  of  $T$  may be regarded as a  $G$ -lattice. Conversely, for a given  $G$ -lattice  $M$ , there exists an algebraic  $k$ -torus  $T$  which splits over  $L$  such that  $X(T)$  is isomorphic to  $M$  as a  $G$ -lattice.

The invariant field  $L(M)^G$  of  $L(M)$  under the action of  $G$  may be identified with the function field of the algebraic  $k$ -torus  $T$ . Note that the field  $L(M)^G$  is always  $k$ -unirational (see [Vos98, page 40, Example 21]). Tori of dimension  $n$  over  $k$  correspond bijectively to the elements of the set  $H^1(\mathcal{G}, \text{GL}(n, \mathbb{Z}))$  where  $\mathcal{G} = \text{Gal}(k_s/k)$  since  $\text{Aut}(\mathbb{G}_m^n) = \text{GL}(n, \mathbb{Z})$ . The  $k$ -torus  $T$  of dimension  $n$  is determined uniquely by the integral representation  $h : \mathcal{G} \rightarrow \text{GL}(n, \mathbb{Z})$  up to conjugacy, and the group  $h(\mathcal{G})$  is a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  (see [Vos98, page 57, Section 4.9]).

Let  $K/k$  be a separable field extension of degree  $n$  and  $L/k$  be the Galois closure of  $K/k$ . Let  $G = \text{Gal}(L/k)$  and  $H = \text{Gal}(L/K)$ . The Galois group  $G$  may be regarded as a transitive subgroup of the symmetric group  $S_n$  of degree  $n$ . Let  $R_{K/k}^{(1)}(\mathbb{G}_m)$  be the norm one torus of  $K/k$ , i.e. the kernel of the norm map  $R_{K/k}(\mathbb{G}_m) \rightarrow \mathbb{G}_m$  where  $R_{K/k}$  is the Weil restriction (see [Vos98, page 37, Section 3.12]). The norm one torus  $R_{K/k}^{(1)}(\mathbb{G}_m)$  has the Chevalley module  $J_{G/H}$  as its character module and the field  $L(J_{G/H})^G$  as its function field where  $J_{G/H} = (I_{G/H})^\circ = \text{Hom}_{\mathbb{Z}}(I_{G/H}, \mathbb{Z})$  is the dual lattice of  $I_{G/H} = \text{Ker } \varepsilon$  and  $\varepsilon : \mathbb{Z}[G/H] \rightarrow \mathbb{Z}$  is the augmentation map (see [Vos98, Section 4.8]). We have the exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G/H] \rightarrow J_{G/H} \rightarrow 0$  and  $\text{rank } J_{G/H} = n - 1$ . Write  $J_{G/H} = \bigoplus_{1 \leq i \leq n-1} \mathbb{Z} x_i$ . Then the action of  $G$  on  $L(J_{G/H}) = L(x_1, \dots, x_{n-1})$  is nothing but (1).

The aim of this paper is to give a birational classification of algebraic  $k$ -tori of dimensions 4 and 5 (cf. [Vos98], [Kun07] and the references therein). It is easy to see that all the 1-dimensional algebraic  $k$ -tori  $T$ , i.e. the trivial torus  $\mathbb{G}_m$  and the norm one torus  $R_{L/k}^{(1)}(\mathbb{G}_m)$  with  $[L : k] = 2$ , are  $k$ -rational.

**Theorem 1.1** (Voskresenskii [Vos67]). *All the 2-dimensional algebraic  $k$ -tori  $T$  are  $k$ -rational. In particular, for any finite subgroups  $G \leq \text{GL}(2, \mathbb{Z})$ ,  $L(x_1, x_2)^G$  is  $k$ -rational.*

There are 13 non-conjugate finite subgroups of  $\text{GL}(2, \mathbb{Z})$ . By Theorem 1.1, we see that for decomposable 3-dimensional  $k$ -tori  $T = T_1 \times T_2$  with  $\dim T_1 = 1$  and  $\dim T_2 = 2$ , the function fields  $L(T) = L(M)^G$  are  $k$ -rational where  $M = M_1 \oplus M_2$  with  $\text{rank } M_1 = 1$  and  $\text{rank } M_2 = 2$ .

Let  $S_n$  (resp.  $A_n$ ,  $D_n$ ,  $C_n$ ) be the symmetric (resp. the alternating, the dihedral, the cyclic) group of degree  $n$  of order  $n!$  (resp.  $n!/2$ ,  $2n$ ,  $n$ ). For  $2 \leq n \leq 4$ , the GAP code  $(n, i, j, k)$  of a finite subgroup  $G$  of  $\text{GL}(n, \mathbb{Z})$  stands for the  $k$ -th  $\mathbb{Z}$ -class of the  $j$ -th  $\mathbb{Q}$ -class of the  $i$ -th crystal system of dimension  $n$  as in [BBNWZ78, Table 1] and [GAP]. There are 73  $\mathbb{Z}$ -classes forming 32  $\mathbb{Q}$ -classes which are classified into 7 crystal systems in  $\text{GL}(3, \mathbb{Z})$ .

A birational classification of the 3-dimensional  $k$ -tori is given by Kunyavskii [Kun90].

**Theorem 1.2** (Kunyavskii [Kun90]). *Let  $L/k$  be a Galois extension and  $G \simeq \text{Gal}(L/k)$  be a finite subgroup of  $\text{GL}(3, \mathbb{Z})$  which acts on  $L(x_1, x_2, x_3)$  via (1). Then  $L(x_1, x_2, x_3)^G$  is not  $k$ -rational if and only if  $G$  is conjugate to one of the 15 groups which are given as in Table 1. Moreover, if  $L(x_1, x_2, x_3)^G$  is not  $k$ -rational, then it is not retract  $k$ -rational.*

Table 1:  $L(M)^G$  not retract  $k$ -rational, rank  $M = 3$ ,  $M$ : indecomposable (15 cases)

${}^tG$ in [Kun90]	GAP code	$G$	${}^tG$ in [Kun90]	GAP code	$G$	${}^tG$ in [Kun90]	GAP code	$G$
$U_1$	(3, 3, 1, 3)	$C_2^2$	$U_6$	(3, 4, 7, 2)	$D_4 \times C_2$	$U_{11}$	(3, 7, 5, 3)	$S_4 \times C_2$
$U_2$	(3, 3, 3, 3)	$C_2^3$	$U_7$	(3, 7, 2, 2)	$A_4 \times C_2$	$U_{12}$	(3, 7, 5, 2)	$S_4 \times C_2$
$U_3$	(3, 4, 4, 2)	$D_4$	$U_8$	(3, 7, 3, 3)	$S_4$	$W_1$	(3, 4, 3, 2)	$C_4 \times C_2$
$U_4$	(3, 4, 6, 3)	$D_4$	$U_9$	(3, 7, 3, 2)	$S_4$	$W_2$	(3, 3, 3, 4)	$C_2^3$
$U_5$	(3, 7, 1, 2)	$A_4$	$U_{10}$	(3, 7, 4, 2)	$S_4$	$W_3$	(3, 7, 2, 3)	$S_4 \times C_2$

If we adopt the action of  $G$  as in (1), we should take the transpose  ${}^tG$  of the matrix group  $G$  as in [Kun90] (cf. Theorem 12.4 in Section 12). For the last statement of Theorem 1.2, see [Kan12, page 25, the fifth paragraph]. We will give an alternative proof of Theorem 1.2 using the algorithms of this paper (see Example 5.3). For  $n = 4$ , some birational invariants are computed by Popov [Pop98].

Let  $T = R_{K/k}^{(1)}(\mathbb{G}_m)$  be the norm one torus defined by  $K/k$ . The rationality problem for norm one tori is investigated by [EM74], [CTS77], [Hür84], [CTS87], [LeB95], [CK00], [LL00], [Flo] and [End11].

**Theorem 1.3.** *Let  $K/k$  be a finite Galois field extension and  $G = \text{Gal}(K/k)$ .*

- (i) (Endo and Miyata [EM74, Theorem 1.5], Saltman [Sal84a, Theorem 3.14])  *$R_{K/k}^{(1)}(\mathbb{G}_m)$  is retract  $k$ -rational if and only if all the Sylow subgroups of  $G$  are cyclic.*
- (ii) (Endo and Miyata [EM74, Theorem 2.3])  *$R_{K/k}^{(1)}(\mathbb{G}_m)$  is stably  $k$ -rational if and only if  $G = C_m$  or  $G = C_n \times \langle \sigma, \tau \mid \sigma^k = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$  where  $d \geq 1, k \geq 3, n, k$ : odd, and  $\gcd\{n, k\} = 1$ .*

**Theorem 1.4** (Endo [End11, Theorem 2.1]). *Let  $K/k$  be a finite non-Galois, separable field extension and  $L/k$  be the Galois closure of  $K/k$ . Assume that the Galois group of  $L/k$  is nilpotent. Then the norm one torus  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is not retract  $k$ -rational.*

**Theorem 1.5** (Endo [End11, Theorem 3.1]). *Let  $K/k$  be a finite non-Galois, separable field extension and  $L/k$  be the Galois closure of  $K/k$ . Let  $G = \text{Gal}(L/k)$  and  $H = \text{Gal}(L/K) \leq G$ . Assume that all the Sylow subgroups of  $G$  are cyclic. Then the norm one torus  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is retract  $k$ -rational, and the following conditions are equivalent:*

- (i)  *$R_{K/k}^{(1)}(\mathbb{G}_m)$  is stably  $k$ -rational;*
- (ii)  *$G = D_n$  with  $n$  odd ( $n \geq 3$ ) or  $G = C_m \times D_n$  where  $m, n$  are odd,  $m, n \geq 3$ ,  $\gcd\{m, n\} = 1$ , and  $H \leq D_n$  is of order 2;*
- (iii)  *$H = C_2$  and  $G \simeq C_r \rtimes H$ ,  $r \geq 3$  odd, where  $H$  acts non-trivially on  $C_r$ .*

**Theorem 1.6** (Colliot-Thélène and Sansuc [CTS87, Proposition 9.1], [LeB95, Theorem 3.1], [CK00, Proposition 0.2], [LL00], Endo [End11, Theorem 4.1], see also [End11, Remark 4.2 and Theorem 4.3]). *Let  $K/k$  be a non-Galois separable field extension of degree  $n$  and  $L/k$  be the Galois closure of  $K/k$ . Assume that  $\text{Gal}(L/k) = S_n$ ,  $n \geq 2$ , and  $\text{Gal}(L/K) = S_{n-1}$  is the stabilizer of one of the letters in  $S_n$ .*

- (i)  *$R_{K/k}^{(1)}(\mathbb{G}_m)$  is retract  $k$ -rational if and only if  $n$  is a prime;*
- (ii)  *$R_{K/k}^{(1)}(\mathbb{G}_m)$  is (stably)  $k$ -rational if and only if  $n = 2, 3$ .*

Let  $[R_{K/k}^{(1)}(\mathbb{G}_m)]^{(t)}$  be the product of  $t$  copies of the norm one torus  $R_{K/k}^{(1)}(\mathbb{G}_m)$ .

**Theorem 1.7** (Endo [End11, Theorem 4.4]). *Let  $K/k$  be a non-Galois separable field extension of degree  $n$  and  $L/k$  be the Galois closure of  $K/k$ . Assume that  $\text{Gal}(L/k) = A_n$ ,  $n \geq 3$ , and  $\text{Gal}(L/K) = A_{n-1}$  is the stabilizer of one of the letters in  $A_n$ .*

- (i)  *$R_{K/k}^{(1)}(\mathbb{G}_m)$  is retract  $k$ -rational if and only if  $n$  is a prime.*
- (ii) *For some positive integer  $t$ ,  $[R_{K/k}^{(1)}(\mathbb{G}_m)]^{(t)}$  is stably  $k$ -rational if and only if  $n = 3, 5$ .*

The first main result of this paper is a birational classification of the algebraic  $k$ -tori of dimension 4. There are 710  $\mathbb{Z}$ -classes forming 227  $\mathbb{Q}$ -classes which are classified into 33 crystal systems in  $\text{GL}(4, \mathbb{Z})$ .

**Theorem 1.8** (Birational classification of the algebraic  $k$ -tori of dimension 4). *Let  $L/k$  be a Galois extension and  $G \simeq \text{Gal}(L/k)$  be a finite subgroup of  $\text{GL}(4, \mathbb{Z})$  which acts on  $L(x_1, x_2, x_3, x_4)$  via (1).*

- (i)  *$L(x_1, x_2, x_3, x_4)^G$  is stably  $k$ -rational if and only if  $G$  is conjugate to one of the 487 groups which are not in Tables 2, 3 and 4.*
- (ii)  *$L(x_1, x_2, x_3, x_4)^G$  is not stably but retract  $k$ -rational if and only if  $G$  is conjugate to one of the 7 groups which are given as in Table 2.*

(iii)  $L(x_1, x_2, x_3, x_4)^G$  is not retract  $k$ -rational if and only if  $G$  is conjugate to one of the 216 groups which are given as in Tables 3 and 4.

Table 2:  $L(M)^G$  not stably but retract  $k$ -rational, rank  $M = 4$ ,  $M$ : indecomposable (7 cases)

GAP code	$G$	GAP code	$G$	GAP code	$G$	GAP code	$G$
(4, 31, 1, 3)	$F_{20}$	(4, 31, 2, 2)	$C_2 \times F_{20}$	(4, 31, 5, 2)	$S_5$	(4, 33, 2, 1)	$C_3 \rtimes C_8$
(4, 31, 1, 4)	$F_{20}$	(4, 31, 4, 2)	$S_5$	(4, 31, 7, 2)	$C_2 \times S_5$		

Table 3:  $L(M)^G$  not retract  $k$ -rational,  $M = M_1 \oplus M_2$ ,  
rank  $M_1 = 3$ , rank  $M_2 = 1$ ,  $M_2$ : indecomposable (64 cases)

GAP code	GAP code	GAP code	GAP code	GAP code	GAP code	GAP code	GAP code
(4, 4, 3, 6)	(4, 6, 2, 4)	(4, 7, 7, 2)	(4, 13, 1, 3)	(4, 13, 7, 8)	(4, 24, 3, 5)	(4, 25, 3, 2)	(4, 25, 7, 4)
(4, 4, 4, 4)	(4, 6, 2, 8)	(4, 12, 2, 4)	(4, 13, 2, 4)	(4, 13, 8, 3)	(4, 24, 4, 3)	(4, 25, 3, 4)	(4, 25, 8, 2)
(4, 4, 4, 6)	(4, 6, 2, 9)	(4, 12, 3, 7)	(4, 13, 3, 4)	(4, 13, 8, 4)	(4, 24, 4, 5)	(4, 25, 4, 4)	(4, 25, 8, 4)
(4, 5, 1, 9)	(4, 6, 3, 3)	(4, 12, 4, 6)	(4, 13, 4, 3)	(4, 13, 9, 3)	(4, 24, 5, 3)	(4, 25, 5, 2)	(4, 25, 9, 4)
(4, 5, 2, 4)	(4, 6, 3, 6)	(4, 12, 4, 8)	(4, 13, 5, 3)	(4, 13, 10, 3)	(4, 24, 5, 5)	(4, 25, 5, 4)	(4, 25, 10, 2)
(4, 5, 2, 7)	(4, 7, 3, 2)	(4, 12, 4, 9)	(4, 13, 6, 3)	(4, 13, 6, 3)	(4, 24, 1, 5)	(4, 25, 1, 2)	(4, 25, 6, 2)
(4, 6, 1, 4)	(4, 7, 4, 3)	(4, 12, 5, 6)	(4, 13, 7, 6)	(4, 24, 2, 3)	(4, 25, 1, 4)	(4, 25, 6, 4)	(4, 25, 11, 2)
(4, 6, 1, 8)	(4, 7, 5, 2)	(4, 12, 5, 7)	(4, 13, 7, 7)	(4, 24, 2, 5)	(4, 25, 2, 4)	(4, 25, 7, 2)	(4, 25, 11, 4)

Table 4:  $L(M)^G$  not retract  $k$ -rational, rank  $M = 4$ ,  $M$ : indecomposable (152 cases)

GAP code	GAP code	GAP code	GAP code	GAP code	GAP code	GAP code	GAP code	GAP code
(4, 5, 1, 12)	(4, 12, 4, 12)	(4, 13, 9, 4)	(4, 19, 3, 2)	(4, 24, 2, 4)	(4, 25, 9, 5)	(4, 29, 8, 1)	(4, 32, 12, 2)	(4, 33, 1, 1)
(4, 5, 2, 5)	(4, 12, 5, 8)	(4, 13, 9, 5)	(4, 19, 4, 3)	(4, 24, 2, 6)	(4, 25, 10, 3)	(4, 29, 8, 2)	(4, 32, 13, 3)	(4, 33, 3, 1)
(4, 5, 2, 8)	(4, 12, 5, 9)	(4, 13, 10, 4)	(4, 19, 4, 4)	(4, 24, 4, 4)	(4, 25, 10, 5)	(4, 29, 9, 1)	(4, 32, 13, 4)	(4, 33, 4, 1)
(4, 5, 2, 9)	(4, 12, 5, 10)	(4, 13, 10, 5)	(4, 19, 5, 2)	(4, 24, 5, 4)	(4, 25, 11, 3)	(4, 32, 1, 2)	(4, 32, 14, 3)	(4, 33, 5, 1)
(4, 6, 1, 6)	(4, 12, 5, 11)	(4, 18, 1, 3)	(4, 19, 6, 2)	(4, 24, 5, 6)	(4, 25, 11, 5)	(4, 32, 2, 2)	(4, 32, 14, 4)	(4, 33, 6, 1)
(4, 6, 1, 11)	(4, 13, 1, 5)	(4, 18, 2, 4)	(4, 22, 1, 1)	(4, 25, 1, 3)	(4, 29, 1, 1)	(4, 32, 3, 2)	(4, 32, 15, 2)	(4, 33, 7, 1)
(4, 6, 2, 6)	(4, 13, 2, 5)	(4, 18, 2, 5)	(4, 22, 2, 1)	(4, 25, 2, 3)	(4, 29, 1, 2)	(4, 32, 4, 2)	(4, 32, 16, 2)	(4, 33, 8, 1)
(4, 6, 2, 10)	(4, 13, 3, 5)	(4, 18, 3, 5)	(4, 22, 3, 1)	(4, 25, 2, 5)	(4, 29, 2, 1)	(4, 32, 5, 2)	(4, 32, 16, 3)	(4, 33, 9, 1)
(4, 6, 2, 12)	(4, 13, 4, 5)	(4, 18, 3, 6)	(4, 22, 4, 1)	(4, 25, 3, 3)	(4, 29, 3, 1)	(4, 32, 5, 3)	(4, 32, 17, 2)	(4, 33, 10, 1)
(4, 6, 3, 4)	(4, 13, 5, 4)	(4, 18, 3, 7)	(4, 22, 5, 1)	(4, 25, 4, 3)	(4, 29, 3, 2)	(4, 32, 6, 2)	(4, 32, 18, 2)	(4, 33, 11, 1)
(4, 6, 3, 7)	(4, 13, 5, 5)	(4, 18, 4, 4)	(4, 22, 5, 2)	(4, 25, 5, 3)	(4, 29, 3, 3)	(4, 32, 7, 2)	(4, 32, 18, 3)	(4, 33, 12, 1)
(4, 6, 3, 8)	(4, 13, 6, 5)	(4, 18, 4, 5)	(4, 22, 6, 1)	(4, 25, 5, 5)	(4, 29, 4, 1)	(4, 32, 8, 2)	(4, 32, 19, 2)	(4, 33, 13, 1)
(4, 12, 2, 5)	(4, 13, 7, 9)	(4, 18, 5, 5)	(4, 22, 7, 1)	(4, 25, 6, 3)	(4, 29, 4, 2)	(4, 32, 9, 4)	(4, 32, 19, 3)	(4, 33, 14, 1)
(4, 12, 2, 6)	(4, 13, 7, 10)	(4, 18, 5, 6)	(4, 22, 8, 1)	(4, 25, 6, 5)	(4, 29, 5, 1)	(4, 32, 9, 5)	(4, 32, 20, 2)	(4, 33, 14, 2)
(4, 12, 3, 11)	(4, 13, 7, 11)	(4, 18, 5, 7)	(4, 22, 9, 1)	(4, 25, 7, 3)	(4, 29, 6, 1)	(4, 32, 10, 2)	(4, 32, 20, 3)	(4, 33, 15, 1)
(4, 12, 4, 10)	(4, 13, 8, 5)	(4, 19, 1, 2)	(4, 22, 10, 1)	(4, 25, 8, 3)	(4, 29, 7, 1)	(4, 32, 11, 2)	(4, 32, 21, 2)	(4, 33, 16, 1)
(4, 12, 4, 11)	(4, 13, 8, 6)	(4, 19, 2, 2)	(4, 22, 11, 1)	(4, 25, 9, 3)	(4, 29, 7, 2)	(4, 32, 11, 3)	(4, 32, 21, 3)	

More detailed information of a birational classification of algebraic  $k$ -tori of dimension 4 is given as in Table 7. In Table 7, # on the second column stands for the number of  $\mathbb{Z}$ -classes in each  $\mathbb{Q}$ -classes, and the list  $[s, r, u]$  stands for the number  $s$  (resp.  $r, u$ ) of  $\mathbb{Z}$ -classes whose invariant field  $L(M)^G$  is stably  $k$ -rational (resp. not stably but retract  $k$ -rational, not retract  $k$ -rational) in each  $\mathbb{Q}$ -classes  $(4, i, j)$ . For example,  $[11, 0, 2]$  in the GAP code  $(4, 5, 1)$  in Table 7 means that the 1st  $\mathbb{Q}$ -class of the 5th crystal system of dimension 4 consists of 13  $\mathbb{Z}$ -classes and  $L(M)^G$  is  $k$ -stably rational for 11  $\mathbb{Z}$ -classes of them, and is not retract  $k$ -rational for 2  $\mathbb{Z}$ -classes.

Let  $G(n, i)$  be the  $i$ -th group of order  $n$  in GAP [GAP]. Let  $dTm$  be the  $m$ -th transitive subgroup of  $S_d$  (cf. [BM83] and [GAP]). Let  $F_{20} \simeq C_5 \rtimes C_4$  be the Frobenius group of order 20.

By Theorem 1.8, we have the following theorem.

**Theorem 1.9.** *Let  $K/k$  be a separable field extension of degree 5 and  $L/k$  be the Galois closure of  $K/k$ . Assume that  $G = \text{Gal}(L/k)$  is a transitive subgroup of  $S_5$  which acts on  $L(x_1, x_2, x_3, x_4)$  via (1), and  $H = \text{Gal}(L/K)$  is the stabilizer of one of the letters in  $G$ . Then a birational classification of the norm one torus  $R_{K/k}^{(1)}(G_m)$  is given as in Table 5.*

Table 5:

$G$	$G(n, i)$	GAP code of the $G$ -action on $J_{G/H}$	$L(J_{G/H})^G$ $= L(x_1, x_2, x_3, x_4)^G$
5T1	$C_5$	$G(5, 1)$	(4, 27, 1, 1) stably $k$ -rational
5T2	$D_5$	$G(10, 1)$	(4, 27, 3, 2) stably $k$ -rational
5T3	$F_{20}$	$G(20, 3)$	(4, 31, 1, 3) not stably but retract $k$ -rational
5T4	$A_5$	$G(60, 5)$	(4, 31, 3, 2) stably $k$ -rational
5T5	$S_5$	$G(120, 34)$	(4, 31, 4, 2) not stably but retract $k$ -rational

Theorem 1.9 is already known except for the case of  $A_5$  (see Theorems 1.3, 1.5, 1.6 and 1.7). Stably  $k$ -rationality of  $R_{K/k}^{(1)}(\mathbb{G}_m)$  for the case  $A_5$  is asked by S. Endo in [End11, Remark 4.6]. By Theorems 1.7 and 1.9, we get:

**Corollary 1.10.** *Let  $K/k$  be a non-Galois separable field extension of degree  $n$  and  $L/k$  be the Galois closure of  $K/k$ . Assume that  $\text{Gal}(L/k) = A_n$ ,  $n \geq 3$ , and  $\text{Gal}(L/K) = A_{n-1}$  is the stabilizer of one of the letters in  $A_n$ . Then  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is stably  $k$ -rational if and only if  $n = 3, 5$ .*

For  $n = 5$ , the CARAT code  $(n, i, j)$  of a finite subgroup  $G$  of  $\text{GL}(5, \mathbb{Z})$  stands for the  $j$ -th  $\mathbb{Z}$ -class of the  $i$ -th  $\mathbb{Q}$ -class in  $\text{GL}(5, \mathbb{Z})$  in the CARAT package [Carat] of GAP<sup>1</sup> (see Section 3 for the details of the CARAT code of dimension  $n \leq 6$ ). There are 6079  $\mathbb{Z}$ -classes forming 955  $\mathbb{Q}$ -classes in  $\text{GL}(5, \mathbb{Z})$ .

The second main result of this paper gives a birational classification of the algebraic  $k$ -tori of dimension 5. We will display Tables 11 to 15 of Theorem 1.11 in Section 13.

**Theorem 1.11** (Birational classification of the algebraic  $k$ -tori of dimension 5). *Let  $L/k$  be a Galois extension and  $G \simeq \text{Gal}(L/k)$  be a finite subgroup of  $\text{GL}(5, \mathbb{Z})$  which acts on  $L(x_1, x_2, x_3, x_4, x_5)$  via (1).*

- (i)  $L(x_1, x_2, x_3, x_4, x_5)^G$  is stably  $k$ -rational if and only if  $G$  is conjugate to one of the 3051 groups which are not in Tables 11, 12, 13, 14 and 15.
- (ii)  $L(x_1, x_2, x_3, x_4, x_5)^G$  is not stably but retract  $k$ -rational if and only if  $G$  is conjugate to one of the 25 groups which are given as in Table 11.
- (iii)  $L(x_1, x_2, x_3, x_4, x_5)^G$  is not retract  $k$ -rational if and only if  $G$  is conjugate to one of the 3003 groups which are given as in Tables 12, 13, 14 and 15.

**Remark 1.12.** For the 25 groups  $G$  as in Theorem 1.11 (ii), the corresponding  $G$ -lattices  $M$  are decomposable  $M \simeq M_1 \oplus M_2$  where  $M_1$  is a  $G/N$ -lattice of rank 4,  $N = \{\sigma \in G \mid \sigma(v) = v \text{ for any } v \in M_1\}$  and  $G/N$  is one of the 7 groups as in Theorem 1.8 (ii) (Table 2) (see Example 4.12). In particular, if  $M$  is an indecomposable  $G$ -lattice of rank 5, then  $L(M)^G$  is either stably  $k$ -rational or not retract  $k$ -rational.

More detailed information of a birational classification of algebraic  $k$ -tori of dimension 5 is given as in Table 16 (see also the explanation of Table 7 above).

**Theorem 1.13.** *Let  $K/k$  be a separable field extension of degree 6 and  $L/k$  be the Galois closure of  $K/k$ . Assume that  $G = \text{Gal}(L/k)$  is a transitive subgroup of  $S_6$  which acts on  $L(x_1, x_2, x_3, x_4, x_5)$  via (1), and  $H = \text{Gal}(L/K)$  is the stabilizer of one of the letters in  $G$ . Then a birational classification of the norm one torus  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is given as in Table 6.*

Table 6:

$G$	$G(n, i)$	CARAT code of the $G$ -action on $J_{G/H}$	$L(J_{G/H})^G$ $= L(x_1, x_2, x_3, x_4, x_5)^G$
6T1	$C_6$	$G(6, 2)$	(5, 461, 4)
6T2	$S_3$	$G(6, 1)$	(5, 173, 4)
6T3	$D_6$	$G(12, 4)$	(5, 391, 4)
6T4	$A_4$	$G(12, 3)$	(5, 580, 2)
6T5	$C_3 \times S_3$	$G(18, 3)$	(5, 823, 4)
6T6	$C_2 \times A_4$	$G(24, 13)$	(5, 606, 2)
6T7	$S_4$	$G(24, 12)$	(5, 607, 2)
6T8	$S_4$	$G(24, 12)$	(5, 608, 2)
6T9	$S_3^2$	$G(36, 10)$	(5, 855, 6)
6T10	$C_3^2 \rtimes C_4$	$G(36, 9)$	(5, 853, 5)
6T11	$C_2 \times S_4$	$G(48, 48)$	(5, 623, 2)
6T12	$A_5$	$G(60, 5)$	(5, 952, 1)
6T13	$S_3^2 \rtimes C_2$	$G(72, 40)$	(5, 892, 4)
6T14	$S_5$	$G(120, 34)$	(5, 947, 1)
6T15	$A_6$	$G(360, 118)$	(5, 951, 1)
6T16	$S_6$	$G(720, 763)$	(5, 953, 1)

In Theorems 1.8, 1.9, 1.11 and 1.13, we do not know whether the field  $L(M)^G$  is  $k$ -rational when the field is stably  $k$ -rational except for few cases (see [Vos98, Chapter 2]).

<sup>1</sup>The generators and some information about the groups  $G \leq \text{GL}(5, \mathbb{Z})$  for the CARAT code  $(5, i, j)$  are available at the second-named author's web page <http://www.math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/>.

Table 7: birational classification of the algebraic  $k$ -tori of dimension 4

GAP code	#	$[s, r, u]$	$G(n, i)$	GAP code	#	$[s, r, u]$	$G(n, i)$
(4, 1, 1)	1	[1,0,0]	$G(1, 1) \quad \{1\}$	(4, 14, 8)	6	[6,0,0]	$G(12, 4) \quad D_6$
(4, 1, 2)	1	[1,0,0]	$G(2, 1) \quad C_2$	(4, 14, 9)	6	[6,0,0]	$G(12, 4) \quad D_6$
(4, 2, 1)	2	[2,0,0]	$G(2, 1) \quad C_2$	(4, 14, 10)	6	[6,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 2, 2)	2	[2,0,0]	$G(2, 1) \quad C_2$	(4, 15, 1)	2	[2,0,0]	$G(12, 5) \quad C_6 \times C_2$
(4, 2, 3)	2	[2,0,0]	$G(4, 2) \quad C_2^2$	(4, 15, 2)	2	[2,0,0]	$G(12, 5) \quad C_6 \times C_2$
(4, 3, 1)	3	[3,0,0]	$G(2, 1) \quad C_2$	(4, 15, 3)	2	[2,0,0]	$G(12, 5) \quad C_6 \times C_2$
(4, 3, 2)	3	[3,0,0]	$G(4, 2) \quad C_2^2$	(4, 15, 4)	2	[2,0,0]	$G(12, 4) \quad D_6$
(4, 4, 1)	6	[6,0,0]	$G(4, 2) \quad C_2^2$	(4, 15, 5)	4	[4,0,0]	$G(12, 4) \quad D_6$
(4, 4, 2)	7	[7,0,0]	$G(4, 2) \quad C_2^2$	(4, 15, 6)	2	[2,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 4, 3)	6	[5,0,1]	$G(4, 2) \quad C_2^2$	(4, 15, 7)	4	[4,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 4, 4)	6	[4,0,2]	$G(8, 5) \quad C_2^3$	(4, 15, 8)	2	[2,0,0]	$G(24, 15) \quad C_6 \times C_2^2$
(4, 5, 1)	13	[11,0,2]	$G(4, 2) \quad C_2^2$	(4, 15, 9)	2	[2,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 5, 2)	9	[4,0,5]	$G(8, 5) \quad C_2^3$	(4, 15, 10)	4	[4,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 6, 1)	12	[8,0,4]	$G(8, 5) \quad C_2^3$	(4, 15, 11)	2	[2,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 6, 2)	12	[6,0,6]	$G(8, 5) \quad C_2^3$	(4, 15, 12)	2	[2,0,0]	$G(48, 51) \quad C_2^3 \times S_3$
(4, 6, 3)	8	[3,0,5]	$G(16, 14) \quad C_2^4$	(4, 16, 1)	3	[3,0,0]	$G(8, 3) \quad D_4$
(4, 7, 1)	2	[2,0,0]	$G(4, 1) \quad C_4$	(4, 17, 1)	3	[3,0,0]	$G(6, 1) \quad S_3$
(4, 7, 2)	2	[2,0,0]	$G(4, 1) \quad C_4$	(4, 17, 2)	2	[2,0,0]	$G(12, 4) \quad D_6$
(4, 7, 3)	2	[1,0,1]	$G(8, 2) \quad C_4 \times C_2$	(4, 18, 1)	3	[2,0,1]	$G(8, 2) \quad C_4 \times C_2$
(4, 7, 4)	4	[3,0,1]	$G(8, 3) \quad D_4$	(4, 18, 2)	5	[3,0,2]	$G(16, 3) \quad (C_4 \times C_2) \times C_2$
(4, 7, 5)	2	[1,0,1]	$G(8, 3) \quad D_4$	(4, 18, 3)	7	[4,0,3]	$G(16, 3) \quad (C_4 \times C_2) \times C_2$
(4, 7, 6)	2	[2,0,0]	$G(8, 3) \quad D_4$	(4, 18, 4)	5	[3,0,2]	$G(16, 11) \quad C_2 \times D_4$
(4, 7, 7)	2	[1,0,1]	$G(16, 11) \quad C_2 \times D_4$	(4, 18, 5)	7	[4,0,3]	$G(32, 27) \quad C_2^4 \times C_2$
(4, 8, 1)	2	[2,0,0]	$G(3, 1) \quad C_3$	(4, 19, 1)	2	[1,0,1]	$G(16, 4) \quad C_4 \times C_4$
(4, 8, 2)	2	[2,0,0]	$G(6, 2) \quad C_6$	(4, 19, 2)	2	[1,0,1]	$G(16, 2) \quad C_4^2$
(4, 8, 3)	3	[3,0,0]	$G(6, 1) \quad S_3$	(4, 19, 3)	2	[1,0,1]	$G(32, 25) \quad C_4 \times D_4$
(4, 8, 4)	3	[3,0,0]	$G(6, 1) \quad S_3$	(4, 19, 4)	4	[2,0,2]	$G(32, 28) \quad (C_4 \times C_2^2) \times C_2$
(4, 8, 5)	3	[3,0,0]	$G(12, 4) \quad D_6$	(4, 19, 5)	2	[1,0,1]	$G(32, 34) \quad C_4^2 \times C_2$
(4, 9, 1)	1	[1,0,0]	$G(6, 2) \quad C_6$	(4, 19, 6)	2	[1,0,1]	$G(64, 226) \quad D_4^2$
(4, 9, 2)	1	[1,0,0]	$G(6, 2) \quad C_6$	(4, 20, 1)	1	[1,0,0]	$G(12, 2) \quad C_{12}$
(4, 9, 3)	1	[1,0,0]	$G(12, 5) \quad C_6 \times C_2$	(4, 20, 2)	1	[1,0,0]	$G(12, 2) \quad C_{12}$
(4, 9, 4)	1	[1,0,0]	$G(12, 4) \quad D_6$	(4, 20, 3)	2	[2,0,0]	$G(12, 1) \quad C_3 \times C_4$
(4, 9, 5)	1	[1,0,0]	$G(12, 4) \quad D_6$	(4, 20, 4)	2	[2,0,0]	$G(24, 5) \quad C_4 \times S_3$
(4, 9, 6)	2	[2,0,0]	$G(12, 4) \quad D_6$	(4, 20, 5)	1	[1,0,0]	$G(24, 9) \quad C_{12} \times C_2$
(4, 9, 7)	1	[1,0,0]	$G(24, 14) \quad C_2^2 \times S_3$	(4, 20, 6)	1	[1,0,0]	$G(24, 10) \quad C_3 \times D_4$
(4, 10, 1)	1	[1,0,0]	$G(4, 1) \quad C_4$	(4, 20, 7)	2	[2,0,0]	$G(24, 6) \quad D_{12}$
(4, 11, 1)	1	[1,0,0]	$G(3, 1) \quad C_3$	(4, 20, 8)	1	[1,0,0]	$G(24, 10) \quad C_3 \times D_4$
(4, 11, 2)	1	[1,0,0]	$G(6, 2) \quad C_6$	(4, 20, 9)	2	[2,0,0]	$G(24, 5) \quad C_4 \times S_3$
(4, 12, 1)	7	[7,0,0]	$G(4, 1) \quad C_4$	(4, 20, 10)	2	[2,0,0]	$G(24, 10) \quad C_3 \times D_4$
(4, 12, 2)	6	[3,0,3]	$G(8, 2) \quad C_4 \times C_2$	(4, 20, 11)	2	[2,0,0]	$G(24, 6) \quad D_{12}$
(4, 12, 3)	13	[11,0,2]	$G(8, 3) \quad D_4$	(4, 20, 12)	4	[4,0,0]	$G(24, 8) \quad (C_6 \times C_2) \times C_2$
(4, 12, 4)	13	[7,0,6]	$G(8, 3) \quad D_4$	(4, 20, 13)	4	[4,0,0]	$G(24, 8) \quad (C_6 \times C_2) \times C_2$
(4, 12, 5)	11	[5,0,6]	$G(16, 11) \quad C_2 \times D_4$	(4, 20, 14)	1	[1,0,0]	$G(24, 7) \quad C_2 \times (C_3 \times C_4)$
(4, 13, 1)	6	[4,0,2]	$G(8, 2) \quad C_4 \times C_2$	(4, 20, 15)	1	[1,0,0]	$G(48, 35) \quad C_2 \times C_4 \times S_3$
(4, 13, 2)	6	[4,0,2]	$G(8, 2) \quad C_4 \times C_2$	(4, 20, 16)	2	[2,0,0]	$G(48, 38) \quad D_4 \times S_3$
(4, 13, 3)	6	[4,0,2]	$G(8, 3) \quad D_4$	(4, 20, 17)	2	[2,0,0]	$G(48, 38) \quad D_4 \times S_3$
(4, 13, 4)	6	[4,0,2]	$G(8, 3) \quad D_4$	(4, 20, 18)	1	[1,0,0]	$G(48, 45) \quad C_6 \times D_4$
(4, 13, 5)	5	[2,0,3]	$G(16, 10) \quad C_4 \times C_2^2$	(4, 20, 19)	1	[1,0,0]	$G(48, 36) \quad C_2 \times D_{12}$
(4, 13, 6)	6	[4,0,2]	$G(16, 11) \quad C_2 \times D_4$	(4, 20, 20)	4	[4,0,0]	$G(48, 38) \quad D_4 \times S_3$
(4, 13, 7)	12	[6,0,6]	$G(16, 11) \quad C_2 \times D_4$	(4, 20, 21)	2	[2,0,0]	$G(48, 43) \quad C_2 \times ((C_6 \times C_2) \rtimes C_2)$
(4, 13, 8)	6	[2,0,4]	$G(16, 11) \quad C_2 \times D_4$	(4, 20, 22)	1	[1,0,0]	$G(96, 209) \quad C_2 \times S_3 \times D_4$
(4, 13, 9)	5	[2,0,3]	$G(16, 11) \quad C_2 \times D_4$	(4, 21, 1)	2	[2,0,0]	$G(6, 2) \quad C_6$
(4, 13, 10)	5	[2,0,3]	$G(32, 46) \quad C_2^2 \times D_4$	(4, 21, 2)	2	[2,0,0]	$G(12, 5) \quad C_6 \times C_2$
(4, 14, 1)	4	[4,0,0]	$G(6, 2) \quad C_6$	(4, 21, 3)	4	[4,0,0]	$G(12, 4) \quad D_6$
(4, 14, 2)	4	[4,0,0]	$G(6, 2) \quad C_6$	(4, 21, 4)	2	[2,0,0]	$G(24, 14) \quad C_2^2 \times S_3$
(4, 14, 3)	8	[8,0,0]	$G(6, 1) \quad S_3$	(4, 22, 1)	2	[1,0,1]	$G(9, 2) \quad C_3^2$
(4, 14, 4)	4	[4,0,0]	$G(12, 5) \quad C_6 \times C_2$	(4, 22, 2)	2	[1,0,1]	$G(18, 5) \quad C_6 \times C_3$
(4, 14, 5)	4	[4,0,0]	$G(12, 4) \quad D_6$	(4, 22, 3)	3	[2,0,1]	$G(18, 3) \quad C_3 \times S_3$
(4, 14, 6)	6	[6,0,0]	$G(12, 4) \quad D_6$	(4, 22, 4)	3	[2,0,1]	$G(18, 3) \quad C_3 \times S_3$
(4, 14, 7)	6	[6,0,0]	$G(12, 4) \quad D_6$	(4, 22, 5)	5	[3,0,2]	$G(18, 4) \quad C_2^2 \times C_2$

Table 7 (continued): birational classification of the algebraic  $k$ -tori of dimension 4

GAP code	#	$[s, r, u]$	$G(n, i)$	GAP code	#	$[s, r, u]$	$G(n, i)$		
(4, 22, 6)	3	[2,0,1]	$G(36, 12)$	$C_6 \times S_3$	(4, 30, 5)	1	[1,0,0]	$G(36, 6)$	$C_3 \times (C_3 \rtimes C_4)$
(4, 22, 7)	3	[2,0,1]	$G(36, 13)$	$C_2 \times (C_3^2 \rtimes C_2)$	(4, 30, 6)	1	[1,0,0]	$G(48, 38)$	$D_4 \times S_3$
(4, 22, 8)	4	[3,0,1]	$G(36, 10)$	$S_3^2$	(4, 30, 7)	1	[1,0,0]	$G(72, 30)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$
(4, 22, 9)	5	[4,0,1]	$G(36, 10)$	$S_3^2$	(4, 30, 8)	1	[1,0,0]	$G(72, 23)$	$(C_6 \times S_3) \rtimes C_2$
(4, 22, 10)	4	[3,0,1]	$G(36, 10)$	$S_3^2$	(4, 30, 9)	1	[1,0,0]	$G(72, 21)$	$(C_3 \times (C_3 \rtimes C_4)) \rtimes C_2$
(4, 22, 11)	4	[3,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(4, 30, 10)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$
(4, 23, 1)	1	[1,0,0]	$G(18, 5)$	$C_6 \times C_3$	(4, 30, 11)	1	[1,0,0]	$G(144, 136)$	$(C_2 \times (C_3^2 \rtimes C_4)) \rtimes C_2$
(4, 23, 2)	1	[1,0,0]	$G(36, 14)$	$C_6^2$	(4, 30, 12)	2	[2,0,0]	$G(144, 115)$	$(C_2 \times (C_3^2 \rtimes C_4)) \rtimes C_2$
(4, 23, 3)	1	[1,0,0]	$G(36, 12)$	$C_6 \times S_3$	(4, 30, 13)	1	[1,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$
(4, 23, 4)	1	[1,0,0]	$G(36, 12)$	$C_6 \times S_3$	(4, 31, 1)	4	[2,2,0]	$G(20, 3)$	$C_5 \rtimes C_4$
(4, 23, 5)	2	[2,0,0]	$G(36, 13)$	$C_2 \times (C_3^2 \rtimes C_2)$	(4, 31, 2)	2	[1,1,0]	$G(40, 12)$	$C_2 \times (C_5 \rtimes C_4)$
(4, 23, 6)	2	[2,0,0]	$G(36, 12)$	$C_6 \times S_3$	(4, 31, 3)	2	[2,0,0]	$G(60, 5)$	$A_5$
(4, 23, 7)	1	[1,0,0]	$G(72, 48)$	$C_2 \times C_6 \times S_3$	(4, 31, 4)	2	[1,1,0]	$G(120, 34)$	$S_5$
(4, 23, 8)	1	[1,0,0]	$G(72, 49)$	$C_2^2 \times (C_3^2 \rtimes C_2)$	(4, 31, 5)	2	[1,1,0]	$G(120, 34)$	$S_5$
(4, 23, 9)	2	[2,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(4, 31, 6)	2	[2,0,0]	$G(120, 35)$	$C_2 \times A_5$
(4, 23, 10)	2	[2,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(4, 31, 7)	2	[1,1,0]	$G(240, 189)$	$C_2 \times S_5$
(4, 23, 11)	1	[1,0,0]	$G(144, 192)$	$C_2^2 \times S_3^2$	(4, 32, 1)	2	[1,0,1]	$G(8, 4)$	$Q_8$
(4, 24, 1)	6	[5,0,1]	$G(12, 3)$	$A_4$	(4, 32, 2)	2	[1,0,1]	$G(16, 6)$	$C_8 \times C_2$
(4, 24, 2)	6	[2,0,4]	$G(24, 13)$	$C_2 \times A_4$	(4, 32, 3)	2	[1,0,1]	$G(16, 8)$	$QD_8$
(4, 24, 3)	6	[5,0,1]	$G(24, 12)$	$S_4$	(4, 32, 4)	2	[1,0,1]	$G(16, 13)$	$(C_4 \times C_2) \rtimes C_2$
(4, 24, 4)	6	[3,0,3]	$G(24, 12)$	$S_4$	(4, 32, 5)	3	[1,0,2]	$G(24, 3)$	$SL(2, 3)$
(4, 24, 5)	6	[2,0,4]	$G(48, 48)$	$C_2 \times S_4$	(4, 32, 6)	2	[1,0,1]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$
(4, 25, 1)	5	[2,0,3]	$G(24, 13)$	$C_2 \times A_4$	(4, 32, 7)	2	[1,0,1]	$G(32, 7)$	$(C_8 \times C_2) \rtimes C_2$
(4, 25, 2)	5	[2,0,3]	$G(24, 13)$	$C_2 \times A_4$	(4, 32, 8)	2	[1,0,1]	$G(32, 11)$	$(C_4 \times C_4) \rtimes C_2$
(4, 25, 3)	5	[2,0,3]	$G(24, 12)$	$S_4$	(4, 32, 9)	5	[3,0,2]	$G(32, 6)$	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$
(4, 25, 4)	5	[3,0,2]	$G(24, 12)$	$S_4$	(4, 32, 10)	2	[1,0,1]	$G(32, 49)$	$(C_2 \times D_4) \rtimes C_2$
(4, 25, 5)	5	[1,0,4]	$G(48, 49)$	$C_2^2 \times A_4$	(4, 32, 11)	3	[1,0,2]	$G(48, 29)$	$GL(2, 3)$
(4, 25, 6)	5	[1,0,4]	$G(48, 48)$	$C_2 \times S_4$	(4, 32, 12)	2	[1,0,1]	$G(64, 134)$	$((C_2 \times D_4) \rtimes C_2) \rtimes C_2$
(4, 25, 7)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$	(4, 32, 13)	4	[2,0,2]	$G(64, 32)$	$((C_8 \times C_2) \rtimes C_2) \rtimes C_2$
(4, 25, 8)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$	(4, 32, 14)	4	[2,0,2]	$G(64, 138)$	
(4, 25, 9)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$	(4, 32, 15)	2	[1,0,1]	$G(64, 34)$	
(4, 25, 10)	5	[1,0,4]	$G(48, 48)$	$C_2 \times S_4$	(4, 32, 16)	3	[1,0,2]	$G(96, 204)$	$((C_2 \times D_4) \rtimes C_2) \rtimes C_3$
(4, 25, 11)	5	[1,0,4]	$G(96, 226)$	$C_2^2 \times S_4$	(4, 32, 17)	2	[1,0,1]	$G(128, 928)$	$D_4^2 \rtimes C_2$
(4, 26, 1)	1	[1,0,0]	$G(8, 1)$	$C_8$	(4, 32, 18)	3	[1,0,2]	$G(192, 201)$	
(4, 26, 2)	1	[1,0,0]	$G(16, 7)$	$D_8$	(4, 32, 19)	3	[1,0,2]	$G(192, 1493)$	
(4, 27, 1)	1	[1,0,0]	$G(5, 1)$	$C_5$	(4, 32, 20)	3	[1,0,2]	$G(192, 1494)$	
(4, 27, 2)	1	[1,0,0]	$G(10, 2)$	$C_{10}$	(4, 32, 21)	3	[1,0,2]	$G(384, 5602)$	
(4, 27, 3)	2	[2,0,0]	$G(10, 1)$	$D_5$	(4, 33, 1)	1	[0,0,1]	$G(24, 11)$	$C_3 \times Q_8$
(4, 27, 4)	1	[1,0,0]	$G(20, 4)$	$D_{10}$	(4, 33, 2)	1	[0,1,0]	$G(24, 1)$	$C_3 \rtimes C_8$
(4, 28, 1)	1	[1,0,0]	$G(12, 2)$	$C_{12}$	(4, 33, 3)	1	[0,0,1]	$G(24, 3)$	$SL(2, 3)$
(4, 28, 2)	1	[1,0,0]	$G(24, 6)$	$D_{12}$	(4, 33, 4)	1	[0,0,1]	$G(48, 17)$	$(C_3 \times Q_8) \rtimes C_2$
(4, 29, 1)	3	[1,0,2]	$G(18, 3)$	$C_3 \times S_3$	(4, 33, 5)	1	[0,0,1]	$G(48, 33)$	$SL(2, 3) \rtimes C_2$
(4, 29, 2)	2	[1,0,1]	$G(36, 12)$	$C_6 \times S_3$	(4, 33, 6)	1	[0,0,1]	$G(48, 29)$	$GL(2, 3)$
(4, 29, 3)	5	[2,0,3]	$G(36, 10)$	$S_3^2$	(4, 33, 7)	1	[0,0,1]	$G(72, 25)$	$C_3 \times SL(2, 3)$
(4, 29, 4)	4	[2,0,2]	$G(36, 9)$	$C_3^2 \rtimes C_4$	(4, 33, 8)	1	[0,0,1]	$G(96, 201)$	$(SL(2, 3) \rtimes C_2) \rtimes C_2$
(4, 29, 5)	2	[1,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(4, 33, 9)	1	[0,0,1]	$G(96, 193)$	$GL(2, 3) \rtimes C_2$
(4, 29, 6)	3	[2,0,1]	$G(72, 45)$	$C_2 \times (C_3^2 \rtimes C_4)$	(4, 33, 10)	1	[0,0,1]	$G(96, 67)$	$SL(2, 3) \rtimes C_4$
(4, 29, 7)	4	[2,0,2]	$G(72, 40)$	$S_3^2 \rtimes C_2$	(4, 33, 11)	1	[0,0,1]	$G(144, 125)$	$(C_3 \times SL(2, 3)) \rtimes C_2$
(4, 29, 8)	4	[2,0,2]	$G(72, 40)$	$S_3^2 \rtimes C_2$	(4, 33, 12)	1	[0,0,1]	$G(192, 988)$	$(GL(2, 3) \rtimes C_2) \rtimes C_2$
(4, 29, 9)	3	[2,0,1]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(4, 33, 13)	1	[0,0,1]	$G(288, 860)$	
(4, 30, 1)	1	[1,0,0]	$G(12, 1)$	$C_3 \rtimes C_4$	(4, 33, 14)	2	[0,0,2]	$G(576, 8277)$	
(4, 30, 2)	1	[1,0,0]	$G(24, 5)$	$C_4 \times S_3$	(4, 33, 15)	1	[0,0,1]	$G(576, 8282)$	
(4, 30, 3)	1	[1,0,0]	$G(24, 10)$	$C_3 \times D_4$	(4, 33, 16)	1	[0,0,1]	$G(1152, 157478)$	
(4, 30, 4)	2	[2,0,0]	$G(24, 8)$	$(C_6 \times C_2) \rtimes C_2$					

Let  $M$  be a  $G$ -lattice.  $M$  is called permutation if  $M$  has a  $\mathbb{Z}$ -basis permuted by  $G$ , i.e.  $M \simeq \bigoplus_{1 \leq i \leq m} \mathbb{Z}[G/H_i]$  for some subgroups  $H_1, \dots, H_m$  of  $G$ .  $M$  is called stably permutation if  $M \oplus P \simeq P'$  for some permutation  $G$ -lattices  $P$  and  $P'$ .  $M$  is called invertible if it is a direct summand of a permutation  $G$ -lattice, i.e.  $P \simeq M \oplus M'$  for some permutation  $G$ -lattice  $P$  and a  $G$ -lattice  $M'$ .  $M$  is called coflabby if  $H^1(H, M) = 0$  for any subgroup  $H$  of  $G$ .  $M$  is called flabby if  $\hat{H}^{-1}(H, M) = 0$  for any subgroup  $H$  of  $G$  where  $\hat{H}$  is the Tate cohomology (see Definition 2.4).

**Definition 1.14** (The categories  $C(G)$ ,  $S(G)$ ,  $D(G)$  and  $H^i(G)$ ). Let  $C(G)$  be the category of all  $G$ -lattices. Let  $S(G)$  be the category of all permutation  $G$ -lattices. Let  $D(G)$  be the category of all invertible  $G$ -lattices. Let

$$H^i(G) = \{M \in C(G) \mid \hat{H}^i(H, M) = 0 \text{ for any } H \leq G\} \quad (i = \pm 1)$$

be the category of “ $\hat{H}^i$ -vanish”  $G$ -lattices where  $\hat{H}^i$  is the Tate cohomology (see Section 2). Then one has the inclusions  $S(G) \subset D(G) \subset H^i(G) \subset C(G)$  ( $i = \pm 1$ ).

**Definition 1.15** (The commutative monoid  $C(G)/S(G)$ ). We say that two  $G$ -lattices  $M_1$  and  $M_2$  are similar if there exist permutation  $G$ -lattices  $P_1$  and  $P_2$  such that  $M_1 \oplus P_1 \simeq M_2 \oplus P_2$ . We denote the similarity class of  $M$  by  $[M]$ . The category of similarity classes  $C(G)/S(G)$  becomes a commutative monoid (with respect to the sum  $[M_1] + [M_2] := [M_1 \oplus M_2]$  and the zero  $0 = [P]$  where  $P \in S(G)$ ).

For a  $G$ -lattice  $M$ , there exists a short exact sequence of  $G$ -lattices  $0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$  where  $P$  is permutation and  $F$  is flabby which is called a flasque resolution of  $M$  (see Theorem 2.12). The similarity class  $[F] \in C(G)/S(G)$  of  $F$  is determined uniquely and is called the flabby class of  $M$ . We denote the flabby class  $[F]$  of  $M$  by  $[M]^{fl}$  (see Definition 2.13). We say that  $[M]^{fl}$  is invertible if  $[M]^{fl} = [E]$  for some invertible  $G$ -lattice  $E$ .

The flabby class  $[M]^{fl}$  plays crucial role in the rationality problem for  $L(M)^G$  as follows (see also Voskresenskii’s fundamental book [Vos98, Section 4.6]):

**Theorem 1.16** (Endo and Miyata, Voskresenskii and Saltman). *Let  $L/k$  be a finite Galois extension with Galois group  $G = \text{Gal}(L/k)$  and  $M$  and  $M'$  be  $G$ -lattices.*

- (i) (Endo and Miyata [EM73, Theorem 1.6])  $[M]^{fl} = 0$  if and only if  $L(M)^G$  is stably  $k$ -rational.
- (ii) (Voskresenskii [Vos74, Theorem 2])  $[M]^{fl} = [M']^{fl}$  if and only if  $L(M)^G$  and  $L(M')^G$  are stably isomorphic, i.e. there exist algebraically independent elements  $x_1, \dots, x_m$  over  $L(M)^G$  and  $y_1, \dots, y_n$  over  $L(M')^G$  such that  $L(M)^G(x_1, \dots, x_m) \simeq L(M')^G(y_1, \dots, y_n)$ .
- (iii) (Saltman [Sal84a, Theorem 3.14])  $[M]^{fl}$  is invertible if and only if  $L(M)^G$  is retract  $k$ -rational.

**Theorem 1.17** (Colliot-Thélène and Sansuc [CTS77, Corollaire 1]). *Let  $G$  be a finite group. The following conditions are equivalent:*

- (i)  $[J_G]^{fl}$  is coflabby;
- (ii) any Sylow subgroup of  $G$  is cyclic or generalized quaternion  $Q_{4n}$  of order  $4n$  ( $n \geq 2$ );
- (iii) any abelian subgroup of  $G$  is cyclic;
- (iv)  $H^3(H, \mathbb{Z}) = 0$  for any subgroup  $H$  of  $G$ .

**Theorem 1.18** (Endo and Miyata [EM82, Theorem 2.1]). *Let  $G$  be a finite group. The following conditions are equivalent:*

- (i)  $H^1(G) \cap H^{-1}(G) = D(G)$ , i.e. any flabby and coflabby  $G$ -lattice is invertible;
- (ii)  $[J_G \otimes_{\mathbb{Z}} J_G]^{fl} = [[J_G]^{fl}]^{fl}$  is invertible;
- (iii) any  $p$ -Sylow subgroup of  $G$  is cyclic for odd  $p$  and cyclic or dihedral (including Klein’s four group) for  $p = 2$ .

Note that  $H^1(H, [J_G]^{fl}) = H^3(H, \mathbb{Z})$  for any subgroups  $H$  of  $G$  (see [Vos70, Theorem 7] and [CTS77, Proposition 1]) and  $[J_G]^{fl} = J_G \otimes_{\mathbb{Z}} J_G$  (see [EM82, Section 2]).

It is not difficult to see

$$“M \text{ is permutation}” \Rightarrow “M \text{ is stably permutation}” \Rightarrow “M \text{ is invertible}” \Rightarrow “M \text{ is flabby and coflabby}”.$$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ [M]^{fl} = 0 & \Rightarrow & [M]^{fl} \text{ is invertible.} \end{array}$$

The above implications in each step cannot be reversed. Swan [Swa60] gave an example of  $Q_8$ -lattice  $M$  of rank 8 which is not permutation but stably permutation:  $M \oplus \mathbb{Z} \simeq \mathbb{Z}[Q_8] \oplus \mathbb{Z}$ . This also indicates that the direct sum cancellation fails (see also Theorem 4.6). Colliot-Thélène and Sansuc [CTS77, Remarque R1] and [CTS77, Remarque R4] gave examples of  $S_3$ -lattice  $M$  of rank 4 which is not permutation but stably permutation:  $M \oplus \mathbb{Z} \simeq \mathbb{Z}[S_3/\langle \sigma \rangle] \oplus \mathbb{Z}[S_3/\langle \tau \rangle]$  where  $S_3 = \langle \sigma, \tau \rangle$  (see also Table 8 of Theorem 6.3) and of  $F_{20}$ -lattice  $[J_{F_{20}/C_4}]^{fl}$  of the Chevalley module  $J_{F_{20}/C_4}$  of rank 4 which is not stably permutation but invertible (see also Theorem 1.21 and Theorem 1.25 (ii), (iv) and (v)). By Theorems 1.3 (i), 1.16 (ii) and 1.17, the flabby class  $[J_{Q_8}]^{fl}$  of the Chevalley module  $J_{Q_8}$  of rank 7 is not invertible but flabby and coflabby (we may take  $[J_{Q_8}]^{fl}$  of rank 9, see Example 7.3). The inverse direction of the vertical implication holds if  $M$  is coflabby (see Lemma 2.11).

By using the interpretation as in Theorem 1.16, Theorems 1.3, 1.5, 1.6 and 1.7 may be obtained by the following theorems:



**Theorem 1.19** (Endo and Miyata [EM74, Theorem 1.5]). *Let  $G$  be a finite group. The following conditions are equivalent:*

- (i)  $[J_G]^{fl}$  is invertible;
- (ii) all the Sylow subgroups of  $G$  are cyclic;
- (iii)  $H^{-1}(G) = H^1(G) = D(G)$ , i.e. any flabby (resp. coflabby)  $G$ -lattice is invertible.

**Theorem 1.20** (Endo and Miyata [EM74, Theorem 2.3], see also [CTS77, Proposition 3]). *Let  $G$  be a finite group. The following conditions are equivalent:*

- (i)  $[J_G]^{fl} = 0$ ;
- (ii)  $[J_G]^{fl}$  is of finite order in  $C(G)/S(G)$ ;
- (iii) all the Sylow subgroups of  $G$  are cyclic and  $H^4(G, \mathbb{Z}) = \hat{H}^0(G, \mathbb{Z})$ ;
- (iv)  $G = C_m$  or  $G = C_n \times \langle \sigma, \tau \mid \sigma^k = \tau^{2^d} = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$  where  $d \geq 1, k \geq 3, n, k$ : odd, and  $\gcd\{n, k\} = 1$ ;
- (v)  $G = \langle s, t \mid s^m = t^{2^d} = 1, tst^{-1} = s^r, m$ : odd,  $r^2 \equiv 1 \pmod{m} \rangle$ .

**Theorem 1.21** (Endo [End11, Theorem 3.1]). *Let  $G$  be a non-abelian group whose Sylow subgroups are all cyclic. Let  $H$  be a non-normal subgroup of  $G$  which contains no normal subgroup of  $G$  except  $\{1\}$ . (By Theorem 1.19,  $[J_{G/H}]^{fl}$  is invertible.) The following conditions are equivalent:*

- (i)  $[J_{G/H}]^{fl} = 0$ ;
- (ii)  $[J_{G/H}]^{fl}$  is of finite order in  $C(G)/S(G)$ ;
- (iii)  $G = D_n$  with  $n$  odd ( $n \geq 3$ ) or  $G = C_m \times D_n$  where  $m, n$  are odd,  $m, n \geq 3$ ,  $\gcd\{m, n\} = 1$ , and  $H \leq D_n$  is of order 2;
- (iv)  $H = C_2$  and  $G \simeq C_r \rtimes H$ ,  $r \geq 3$  odd, where  $H$  acts non-trivially on  $C_r$ .

A partial result of Theorem 1.21 was given by Colliot-Thélène and Sansuc [CTS77, Remarque R4].

**Theorem 1.22** (Colliot-Thélène and Sansuc [CTS87, Proposition 9.1], [LeB95, Theorem 3.1], [CK00, Proposition 0.2], [LL00], Endo [End11, Theorem 4.1], see also [End11, Remark 4.2 and Theorem 4.3]). *Let  $n \geq 2$  be an integer.*

- (i)  $[J_{S_n/S_{n-1}}]^{fl}$  is invertible if and only if  $n$  is a prime.
- (ii)  $[J_{S_n/S_{n-1}}]^{fl} = 0$  if and only if  $n = 2, 3$ .
- (iii)  $[J_{S_n/S_{n-1}}]^{fl}$  is of finite order in  $C(G)/S(G)$  if and only if  $n = 2, 3$ .

**Theorem 1.23** (Endo [End11, Theorem 4.5]). *Let  $n \geq 3$  be an integer.*

- (i)  $[J_{A_n/A_{n-1}}]^{fl}$  is invertible if and only if  $n$  is a prime.
- (ii)  $[J_{A_n/A_{n-1}}]^{fl}$  is of finite order in  $C(G)/S(G)$  if and only if  $n = 3, 5$ .

Note that  $[J_{A_3/A_2}]^{fl} = [J_{C_3}]^{fl} = 0$ . By [Dre75, Corollary 3.3],  $[J_{A_5/A_4}]^{fl}$  is of finite order in  $C(G)/S(G)$ . Indeed, we get  $[J_{A_5/A_4}]^{fl} = 0$  (see Theorems 1.8 and 1.9, Table 5, Corollary 1.10 and Theorem 1.25 (i) below).

**Definition 1.24** (The  $G$ -lattice  $M_G$  of a finite subgroup  $G$  of  $\mathrm{GL}(n, \mathbb{Z})$ ). Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$ . The  $G$ -lattice  $M_G$  of rank  $n$  is defined to be the  $G$ -lattice with a  $\mathbb{Z}$ -basis  $\{u_1, \dots, u_n\}$  on which  $G$  acts by  $\sigma(u_i) = \sum_{j=1}^n a_{i,j} u_j$  for any  $\sigma = [a_{i,j}] \in G$ .

A birational classification of the  $k$ -tori of dimensions 4 and 5 (Theorem 1.8 and Theorem 1.11) may be obtained by the following two theorems respectively.

**Theorem 1.25.** *Let  $G$  be a finite subgroup of  $\mathrm{GL}(4, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24.*

- (i)  $[M_G]^{fl} = 0$  if and only if  $G$  is conjugate to one of the 487 groups which are not in Tables 2, 3 and 4.
- (ii)  $[M_G]^{fl}$  is not zero but invertible if and only if  $G$  is conjugate to one of the 7 groups which are given as in Table 2.
- (iii)  $[M_G]^{fl}$  is not invertible if and only if  $G$  is conjugate to one of the 216 groups which are given as in Tables 3 and 4.
- (iv)  $[M_G]^{fl} = 0$  if and only if  $[M_G]^{fl}$  is of finite order in  $C(G)/S(G)$ .
- (v) For the group  $G \simeq S_5$  of the GAP code (4, 31, 5, 2) in (ii), we have

$$-[M_G]^{fl} = [J_{S_5/S_4}]^{fl} \neq 0.$$

- (vi) For the group  $G \simeq F_{20}$  of the GAP code (4, 31, 1, 4) in (ii), we have

$$-[M_G]^{fl} = [J_{F_{20}/C_4}]^{fl} \neq 0.$$

**Theorem 1.26.** *Let  $G$  be a finite subgroup of  $\mathrm{GL}(5, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24.*

- (i)  $[M_G]^{fl} = 0$  if and only if  $G$  is conjugate to one of the 3051 groups which are not in Tables 11, 12, 13, 14 and

15.

(ii)  $[M_G]^{fl}$  is not zero but invertible if and only if  $G$  is conjugate to one of the 25 groups which are given as in Table 11.

(iii)  $[M_G]^{fl}$  is not invertible if and only if  $G$  is conjugate to one of the 3003 groups which are given as in Tables 12, 13, 14 and 15.

(iv)  $[M_G]^{fl} = 0$  if and only if  $[M_G]^{fl}$  is of finite order in  $C(G)/S(G)$ .

**Remark 1.27.** (i) By the interpretation as in Theorem 1.16, Theorem 1.25 (v), (vi) claims that the corresponding two tori  $T$  and  $T'$  of dimension 4 are not stably  $k$ -rational and are not stably birationally isomorphic each other but the torus  $T \times T'$  of dimension 8 itself is stably  $k$ -rational.

(ii) When  $[M]^{fl}$  is invertible, the inverse element of  $[M]^{fl}$  is given by  $-[M]^{fl} = [[M]^{fl}]^{fl}$  (see Lemma 2.15). Hence Theorem 1.25 (v) also claims that  $[[M_G]^{fl}]^{fl} = [J_{S_5/S_4}]^{fl}$  and  $[[J_{S_5/S_4}]^{fl}]^{fl} = [M_G]^{fl}$ .

We will give proofs of Theorem 1.25 and Theorem 1.26 in Section 9 and Section 10 respectively.

By using the algorithms in Section 5, we show the following theorem:

**Theorem** (Theorem 6.2 and Theorem 6.3). *Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24.*

(i) *When  $n \leq 3$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation.*

(ii) *When  $n = 4$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation or the GAP code of  $G$  is one of  $(4, 14, 2, 2), (4, 14, 3, 3), (4, 14, 3, 4), (4, 14, 8, 2)$ .*

*(There are 11 conjugacy classes of subgroups of  $S_4$  and hence 15 flabby and coflabby  $G$ -lattices of rank 4 in total.)*

(iii) *When  $n = 5$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation or the CARAT code of  $G$  is one of  $(5, 218, 4), (5, 911, 4), (5, 918, 4), (5, 931, 4)$ .*

*(There are 19 conjugacy classes of subgroups of  $S_5$  and hence 23 flabby and coflabby  $G$ -lattices of rank 5 in total.)*

(iv) *When  $n = 6$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation or the CARAT code of  $G$  is one of the 50 triples*

(6, 159, 14),	(6, 161, 14),	(6, 161, 28),	(6, 197, 14),	(6, 226, 14),
(6, 226, 40),	(6, 231, 39),	(6, 238, 27),	(6, 246, 21),	(6, 366, 27),
(6, 1087, 20),	(6, 1090, 18),	(6, 1142, 8),	(6, 2043, 4),	(6, 2051, 9),
(6, 2068, 6),	(6, 2069, 6),	(6, 2069, 12),	(6, 2070, 12),	(6, 2079, 14),
(6, 2079, 28),	(6, 2088, 18),	(6, 2105, 12),	(6, 2154, 26),	(6, 2156, 40),
(6, 2156, 80),	(6, 2188, 39),	(6, 2968, 4),	(6, 2969, 4),	(6, 2969, 8),
(6, 2977, 6),	(6, 3068, 7),	(6, 3068, 8),	(6, 3071, 7),	(6, 3071, 8),
(6, 3073, 7),	(6, 3073, 8),	(6, 3073, 15),	(6, 3073, 16),	(6, 3076, 7),
(6, 3076, 8),	(6, 3091, 11),	(6, 3091, 12),	(6, 5210, 14),	(6, 5262, 11),
(6, 5321, 6),	(6, 5421, 6),	(6, 5475, 6),	(6, 5477, 11),	(6, 5487, 11),

*(There are 56 conjugacy classes of subgroups of  $S_6$  and hence 106 flabby and coflabby  $G$ -lattices of rank 6 in total.)*

(v) *When  $n \leq 6$ ,  $M$  is flabby and coflabby if and only if  $M$  is stably permutation. Indeed, flabby and coflabby  $G$ -lattices  $M$  which are not permutation in (ii), (iii), (iv) are stably permutation as in Table 8.*

**Definition 1.28** (Decomposition type). Let  $G$  be a finite group and  $M$  be a  $G$ -lattice. A  $G$ -lattice  $M$  is said to be decomposable if there exist non-trivial  $G$ -lattices  $U_1$  and  $U_2$  such that  $M \simeq U_1 \oplus U_2$ . A  $G$ -lattice is said to be indecomposable if it is not decomposable. When  $M$  decomposes into indecomposable  $G$ -lattices  $M \simeq U_1 \oplus \cdots \oplus U_r$  of rank  $n_1, \dots, n_r$ , we say that a decomposition type  $\mathrm{DT}(M)$  of  $M$  is  $(n_1, \dots, n_r)$ .

For  $n \leq 6$ , the number of  $G$ -lattices  $M_G$  of rank  $n$  for a given decomposition type  $\mathrm{DT}(M_G)$  is as follows:

$\mathrm{DT}(M_G)$	(1)	Total	$\mathrm{DT}(M_G)$	(1, 1)	(2)	Total	$\mathrm{DT}(M_G)$	(1, 1, 1)	(2, 1)	(3)	Total
$\#M_G$	2	2	$\#M_G$	4	9	13	$\#M_G$	8	31	34	73
$\mathrm{DT}(M_G)$	(1, 1, 1, 1)	(2, 1, 1)	(2, 2)	(3, 1)	(4)	Total					
$\#M_G$	16	96	175	128	295	710					
$\mathrm{DT}(M_G)$	(1 <sup>5</sup> )	(2, 1 <sup>3</sup> )	(2 <sup>2</sup> , 1)	(3, 1 <sup>2</sup> )	(3, 2)	(4, 1)	(5)	Total			
$\#M_G$	32	280	1004	442	<b>1480</b>	<b>1400</b>	1452	<b>6090</b> (6079)			

DT( $M_G$ )	(1 <sup>6</sup> )	(2, 1 <sup>4</sup> )	(2 <sup>2</sup> , 1 <sup>2</sup> )	(2 <sup>3</sup> )	(3, 1 <sup>3</sup> )	(3, 2, 1)	(3 <sup>2</sup> )	(4, 1 <sup>2</sup> )	(4, 2)	(5, 1)	(6)	Total
# $M_G$	68	824	4862	6878	1466	<b>10662</b>	<b>4235</b>	<b>5944</b>	21573	<b>9931</b>	18996	<b>85439</b> (85308)

For  $n \leq 4$ , we see that the Krull-Schmidt theorem holds, i.e. if  $M_G \simeq M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $M_i \simeq N_i$  for any  $1 \leq i \leq m$ . However, it turns out that the Krull-Schmidt theorem fails for  $n = 5$ . We split the Krull-Schmidt theorem into the following two parts:

(KS1) If  $M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $\text{rank } M_i = \text{rank } N_i$  for any  $1 \leq i \leq m$ ;

(KS2) If  $M_1 \oplus \cdots \oplus M_m \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_i$  with  $\text{rank } M_i = \text{rank } N_i$  for any  $1 \leq i \leq m$ , then after a suitable renumbering of the  $N_i$ ,  $M_i \simeq N_i$  for any  $1 \leq i \leq m$ .

Krull-Schmidt theorem holds if and only if the conditions (KS1) and (KS2) hold.

**Theorem** (Theorem 4.6). *Let  $G$  be a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24.*

(i) *When  $n \leq 4$ , the Krull-Schmidt theorem holds, i.e. if  $M_G \simeq M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $M_i \simeq N_i$  for any  $1 \leq i \leq m$ .*

(ii) *When  $n = 5$ , (KS2) holds, and the Krull-Schmidt theorem fails if and only if (KS1) fails if and only if the CARAT code of  $G$  is one of the 11 triples*

$$(5, 188, 4), (5, 189, 4), (5, 190, 6), (5, 191, 6), (5, 192, 6), (5, 193, 4), (5, 205, 6), (5, 218, 8), (5, 219, 8), (5, 220, 4), (5, 221, 4).$$

*For the exceptional 11 cases, the decomposition types of  $M_G$  are (3, 2) and (4, 1) and  $G$  is a subgroup of the group  $C_2 \times D_6$  of the CARAT code (5, 205, 6).*

(iii) *When  $n = 6$ , (KS1) fails if and only if the CARAT code of  $G$  is one of the 131 triples*

$$\begin{aligned} &(6, 2013, 8), (6, 2018, 4), (6, 2023, 6), (6, 2024, 6), (6, 2025, 6), (6, 2026, 6), (6, 2033, 6), (6, 2042, 8), (6, 2043, 8), (6, 2044, 4), \\ &(6, 2045, 4), (6, 2048, 5), (6, 2049, 8), (6, 2050, 8), (6, 2051, 8), (6, 2052, 8), (6, 2058, 5), (6, 2059, 5), (6, 2067, 5), (6, 2068, 5), \\ &(6, 2069, 5), (6, 2069, 11), (6, 2070, 9), (6, 2071, 9), (6, 2072, 10), (6, 2072, 11), (6, 2076, 24), (6, 2076, 25), (6, 2077, 24), (6, 2077, 25), \\ &(6, 2078, 24), (6, 2078, 25), (6, 2079, 24), (6, 2079, 25), (6, 2087, 15), (6, 2088, 15), (6, 2089, 17), (6, 2089, 18), (6, 2094, 9), (6, 2102, 24), \\ &(6, 2102, 25), (6, 2105, 9), (6, 2106, 9), (6, 2107, 10), (6, 2107, 11), (6, 2108, 15), (6, 2109, 15), (6, 2110, 17), (6, 2110, 18), (6, 2111, 15), \\ &(6, 2139, 9), \\ &(6, 40, 4), (6, 41, 4), (6, 44, 6), (6, 45, 6), (6, 47, 4), (6, 53, 4), (6, 54, 4), (6, 54, 8), (6, 55, 4), (6, 63, 4), \\ &(6, 64, 6), (6, 65, 4), (6, 66, 6), (6, 67, 6), (6, 75, 4), (6, 75, 8), (6, 76, 8), (6, 76, 12), (6, 77, 8), (6, 77, 12), \\ &(6, 78, 4), (6, 78, 8), (6, 79, 6), (6, 80, 4), (6, 81, 8), (6, 81, 12), (6, 90, 4), (6, 99, 4), (6, 108, 4), (6, 108, 8), \\ &(6, 109, 8), (6, 109, 12), (6, 110, 4), (6, 111, 6), (6, 112, 8), (6, 112, 12), (6, 113, 4), (6, 114, 6), (6, 115, 6), (6, 145, 4), \\ &(6, 2070, 10), (6, 2070, 11), (6, 2071, 10), (6, 2071, 11), (6, 2072, 12), (6, 2072, 13), (6, 2076, 26), (6, 2076, 27), (6, 2077, 26), (6, 2077, 27), \\ &(6, 2078, 26), (6, 2078, 27), (6, 2079, 26), (6, 2079, 27), (6, 2087, 16), (6, 2087, 17), (6, 2088, 16), (6, 2088, 17), (6, 2089, 19), (6, 2089, 20), \\ &(6, 2094, 10), (6, 2094, 11), (6, 2102, 26), (6, 2102, 27), (6, 2105, 10), (6, 2105, 11), (6, 2106, 10), (6, 2106, 11), (6, 2107, 12), (6, 2107, 13), \\ &(6, 2108, 16), (6, 2108, 17), (6, 2109, 16), (6, 2109, 17), (6, 2110, 19), (6, 2110, 20), (6, 2111, 16), (6, 2111, 17), (6, 2139, 10), (6, 2139, 11). \end{aligned}$$

*For the former 51 cases (resp. the latter 80 cases), the decomposition types of  $M_G$  are (3, 2, 1) and (4, 1, 1) (resp. (3, 3) and (5, 1)) and  $G$  is a subgroup of the group  $C_2^2 \times D_6$  of the CARAT code (6, 2139, 9) (resp.  $D_6 \times D_4$  of the CARAT code (6, 145, 4)).*

(iv) *When  $n = 6$ , (KS2) fails if and only if the CARAT code of  $G$  is one of the 18 triples*

$$\begin{aligned} &(6, 2072, 14), (6, 2076, 28), (6, 2077, 28), (6, 2078, 28), (6, 2079, 28), (6, 2089, 21), (6, 2102, 28), (6, 2107, 14), (6, 2110, 21), (6, 2295, 2), \\ &(6, 3045, 3), (6, 3046, 3), (6, 3047, 3), (6, 3052, 5), (6, 3053, 5), (6, 3054, 3), (6, 3061, 5), (6, 3066, 3). \end{aligned}$$

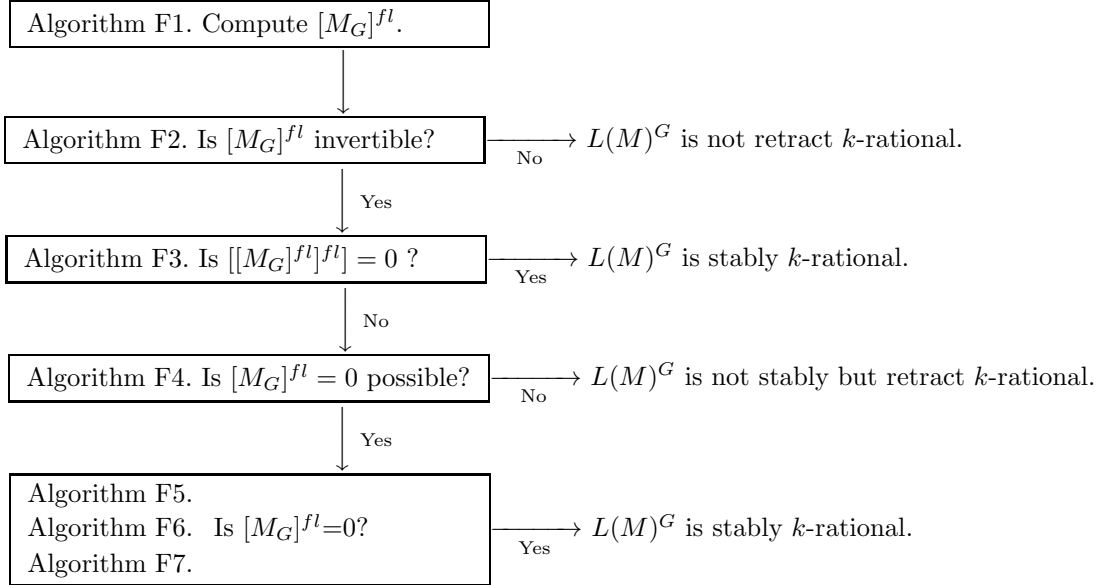
*For the former 10 cases, the decomposition type of  $M_G$  is (4, 2) and  $G$  is the group  $D_6$  of the CARAT code (6, 2295, 2) or a subgroup of the 3 groups  $C_2 \times D_6$  of the CARAT codes (6, 2102, 28), (6, 2107, 14) and (6, 2110, 21). For the latter 8 cases, the decomposition type of  $M_G$  is (5, 1) and  $G$  is a subgroup of the group  $C_2 \times S_5$  of the CARAT code (6, 3054, 3).*

Using the algorithms as in Section 5 and Section 6, we may verify the following isomorphism which gives the smallest example exhibiting the failure of the Krull-Schmidt theorem for permutation  $G$ -lattices (see Section 4 and Dress's paper [Dre73, Proposition 9.6]):

**Proposition** (Proposition 6.7). *Let  $D_6$  be the dihedral group of order 12 and  $\{1\}, C_2^{(1)}, C_2^{(2)}, C_2^{(3)}, C_3, C_2^2, C_6, S_3^{(1)}, S_3^{(2)}$  and  $D_6$  be the conjugacy classes of subgroups of  $D_6$ . Then the following isomorphism of permutation  $D_6$ -lattices holds:*

$$\begin{aligned} & \mathbb{Z}[D_6] \oplus 2\mathbb{Z}[D_6/C_2^2] \oplus \mathbb{Z}[D_6/C_6] \oplus \mathbb{Z}[D_6/S_3^{(1)}] \oplus \mathbb{Z}[D_6/S_3^{(2)}] \\ & \simeq \mathbb{Z}[D_6/C_2^{(1)}] \oplus \mathbb{Z}[D_6/C_2^{(2)}] \oplus \mathbb{Z}[D_6/C_2^{(3)}] \oplus \mathbb{Z}[D_6/C_3] \oplus 2\mathbb{Z}. \end{aligned}$$

The following flow chart presents the structure of the GAP algorithms which will be given in Section 5.



As an application of the method of this paper, we give examples of not retract  $k$ -rational fields which are related to the rationality problem under the finite group action, e.g. Noether's problem (see [HKY11], [Yam12], [HKKi]).

Let  $k$  be a field of char  $k \neq 2$  and  $k(x, y, z)$  be the rational function field over  $k$  with variables  $x, y, z$ . We consider the  $k$ -involution (i.e.  $k$ -automorphism of order 2)

$$\sigma_{a,b,c,d} : x \mapsto -x, \quad y \mapsto \frac{-ax^2 + b}{y}, \quad z \mapsto \frac{-cx^2 + d}{z} \quad (a, b, c, d \in k^\times)$$

on  $k(x, y, z)$  and the rationality problem of  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  over  $k$ , namely whether the fixed field  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational. We see that the fixed field  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -isomorphic to  $k(x, y, z)^{\langle \sigma_{\tau(a), \tau(b), \tau(c), \tau(d)} \rangle}$  for  $\tau \in D_4$  where  $D_4 = \langle (abcd), (ab)(cd) \rangle$  is the permutation group on the set  $\{a, b, c, d\}$  which is isomorphic to the dihedral group of order 8. Let  $m = [k(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) : k]$ . Hence the rationality problem is determined by the following 22 cases:

- (C1)  $m = 1$ ;
- (C2)  $m = 2$ , (1)  $a, b, c \in k^{\times 2}$ ; (2)  $a, b, d \in k^{\times 2}$ ; (3)  $a, c, d \in k^{\times 2}$ ; (4)  $b, c, d \in k^{\times 2}$ ;
- (C3)  $m = 2$ , (1)  $a, b, cd \in k^{\times 2}$ ; (2)  $b, d, ac \in k^{\times 2}$ ; (3)  $d, c, ab \in k^{\times 2}$ ; (4)  $c, a, bd \in k^{\times 2}$ ;
- (C4)  $m = 2$ , (1)  $a, d, bc \in k^{\times 2}$ ; (2)  $b, c, ad \in k^{\times 2}$ ;
- (C5)  $m = 2$ , (1)  $a, bd, cd \in k^{\times 2}$ ; (2)  $b, cd, ac \in k^{\times 2}$ ; (3)  $d, ac, ab \in k^{\times 2}$ ; (4)  $c, ab, bd \in k^{\times 2}$ ;
- (C6)  $m = 2$ ,  $ab, ac, ad \in k^{\times 2}$ ;
- (C7)  $m = 4$ , (1)  $a, b \in k^{\times 2}$ ; (2)  $b, d \in k^{\times 2}$ ; (3)  $d, c \in k^{\times 2}$ ; (4)  $c, a \in k^{\times 2}$ ;
- (C8)  $m = 4$ , (1)  $a, d \in k^{\times 2}$ ; (2)  $b, c \in k^{\times 2}$ ;
- (C9)  $m = 4$ , (1)  $a, bc \in k^{\times 2}$ ; (2)  $b, ad \in k^{\times 2}$ ; (3)  $d, bc \in k^{\times 2}$ ; (4)  $c, ad \in k^{\times 2}$ ;
- (C10)  $m = 4$ , (1)  $a, bd \in k^{\times 2}$ ; (2)  $b, dc \in k^{\times 2}$ ; (3)  $d, ac \in k^{\times 2}$ ; (4)  $c, ab \in k^{\times 2}$ ;
- (5)  $a, cd \in k^{\times 2}$ ; (6)  $b, ac \in k^{\times 2}$ ; (7)  $d, ab \in k^{\times 2}$ ; (8)  $c, bd \in k^{\times 2}$ ;
- (C11)  $m = 4$ , (1)  $a, bcd \in k^{\times 2}$ ; (2)  $b, acd \in k^{\times 2}$ ; (3)  $d, abc \in k^{\times 2}$ ; (4)  $c, abd \in k^{\times 2}$ ;
- (C12)  $m = 4$ , (1)  $ab, cd \in k^{\times 2}$ ; (2)  $bd, ac \in k^{\times 2}$ ;
- (C13)  $m = 4$ , (1)  $ab, ac \in k^{\times 2}$ ; (2)  $bd, ab \in k^{\times 2}$ ; (3)  $cd, bd \in k^{\times 2}$ ; (4)  $ac, cd \in k^{\times 2}$ ;
- (C14)  $m = 4$ ,  $ad, bc \in k^{\times 2}$ ;

- (C15)  $m = 4$ , (1)  $ab, acd \in k^{\times 2}$ ; (2)  $bd, abc \in k^{\times 2}$ ; (3)  $cd, abd \in k^{\times 2}$ ; (4)  $ac, bcd \in k^{\times 2}$ ;  
 (C16)  $m = 4$ , (1)  $ad, abc \in k^{\times 2}$ ; (2)  $bc, abd \in k^{\times 2}$ ;  
 (C17)  $m = 8$ , (1)  $a \in k^{\times 2}$ ; (2)  $b \in k^{\times 2}$ ; (3)  $d \in k^{\times 2}$ ; (4)  $c \in k^{\times 2}$ ;  
 (C18)  $m = 8$ , (1)  $ab \in k^{\times 2}$ ; (2)  $ac \in k^{\times 2}$ ; (3)  $bd \in k^{\times 2}$ ; (4)  $cd \in k^{\times 2}$ ;  
 (C19)  $m = 8$ , (1)  $ad \in k^{\times 2}$ ; (2)  $bc \in k^{\times 2}$ ;  
 (C20)  $m = 8$ , (1)  $abc \in k^{\times 2}$ ; (2)  $bcd \in k^{\times 2}$ ; (3)  $abd \in k^{\times 2}$ ; (4)  $acd \in k^{\times 2}$ ;  
 (C21)  $m = 8$ ,  $abcd \in k^{\times 2}$ ;  
 (C22)  $m = 16$ .

We see that if one of the conditions (C1), (C2), (C3), (C5), (C6), (C7), (C10), (C12), (C13) holds, then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational (see Lemma 11.5).

**Theorem** (Theorem 11.3). *Let  $k$  be a field of char  $k \neq 2$  and  $k(x, y, z)$  be the rational function field over  $k$  with variables  $x, y, z$ . Let  $\sigma_{a,b,c,d}$  be a  $k$ -involution on  $k(x, y, z)$  defined by*

$$\sigma_{a,b,c,d} : x \mapsto -x, \quad y \mapsto \frac{-ax^2 + b}{y}, \quad z \mapsto \frac{-cx^2 + d}{z} \quad (a, b, c, d \in k^\times)$$

and  $m = [k(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) : k]$ .

(i)  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = k(t_1, t_2, t_3, t_4)$  where  $t_1, t_2, t_3, t_4$  satisfy the relation

$$(t_1^2 - a)(t_4^2 - d) = (t_2^2 - b)(t_3^2 - c).$$

(ii)  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -isomorphic to  $k(x, y, z)^{\langle \sigma_{\tau(a), \tau(b), \tau(c), \tau(d)} \rangle}$  for  $\tau \in D_4$  where  $D_4 = \langle (abdc), (ab)(cd) \rangle$  is the permutation group on the set  $\{a, b, c, d\}$  which is isomorphic to the dihedral group of order 8.

(iii) If one of the following conditions holds, then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract  $k$ -rational:

- (C15)  $m = 4$ , (1)  $ab, acd \in k^{\times 2}$ ; (2)  $bd, abc \in k^{\times 2}$ ; (3)  $cd, abd \in k^{\times 2}$ ; (4)  $ac, bcd \in k^{\times 2}$ ;  
 (C16)  $m = 4$ , (1)  $ad, abc \in k^{\times 2}$ ; (2)  $bc, abd \in k^{\times 2}$ ;  
 (C18)  $m = 8$ , (1)  $ab \in k^{\times 2}$ ; (2)  $ac \in k^{\times 2}$ ; (3)  $bd \in k^{\times 2}$ ; (4)  $cd \in k^{\times 2}$ ;  
 (C19)  $m = 8$ , (1)  $ad \in k^{\times 2}$ ; (2)  $bc \in k^{\times 2}$ ;  
 (C20)  $m = 8$ , (1)  $abc \in k^{\times 2}$ ; (2)  $bcd \in k^{\times 2}$ ; (3)  $abd \in k^{\times 2}$ ; (4)  $acd \in k^{\times 2}$ ;  
 (C21)  $m = 8$ ,  $abcd \in k^{\times 2}$ ;  
 (C22)  $m = 16$ .

We do not know whether the field  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational for the cases (C4), (C8), (C9), (C11), (C14), (C17).

We organize this paper as follows. In Section 2, we recall known results and prepare some basic materials, e.g. Galois cohomology, Tate cohomology, flabby resolution of a  $G$ -lattice. We explain the relationship between these materials and the rationality problem of algebraic  $k$ -tori. In Section 3, we explain how to access the GAP code and the CRAT code of a finite subgroup  $G$  of  $\mathrm{GL}(n, \mathbb{Z})$  ( $n \leq 6$ ) of this paper. In Section 4, we show that the Krull-Schmidt theorem for  $G$ -lattices holds when the rank  $\leq 4$ , and fails when the rank is 5 and 6 by using the GAP code and the CARAT code. In Section 5, we give some algorithms to compute a flabby resolution of a  $G$ -lattice effectively in GAP. We also give some algorithms which may determine whether the flabby class of the  $G$ -lattice is invertible (resp. zero) or not. These algorithms enable us to classify the function fields of algebraic  $k$ -tori birationally. In Section 6, we give a classification of all the flabby and coflabby  $G$ -lattices of rank up to 6. In Section 7, we obtain a birational classification of the norm one tori of dimensions 4 and 5. In Section 8, we will give some GAP algorithms for computing the Tate cohomologies. In Sections 9 and 10, we give proofs of Theorem 1.25 and Theorem 1.26 respectively by using the algorithms which is given in Sections 3, 4, 5 and 8. Using the algorithms in Section 5, we will show Theorem 11.3 which provides some examples of not retract  $k$ -rational fields in Section 11. In Section 12, an application of Theorem 11.3 which is related to Noether's problem is given. Tables 11 to 15 in Theorems 1.11 and 1.26 are located in Section 13. We also give detailed information of a birational classification of the algebraic  $k$ -tori of dimension 5 as in Table 16 in Section 13.

**Acknowledgments.** The authors would like to thank Ming-chang Kang for giving them useful and valuable comments. They also would like to thank Shizuo Endo for valuable comments and for fruitful discussions about Section 4. The first-named author wishes to gratefully thank his teacher Yumiko Hironaka for continuous encouragement.

## 2. PRELIMINARIES: TATE COHOMOLOGY AND FLABBY RESOLUTIONS

First we recall the definitions of Galois cohomology and Tate cohomology. See for details Cartan and Eilenberg [CE56, Chapter XII] and Brown [Bro82, Chapter VI].

**Definition 2.1** (*n*-cochains, coboundary homomorphisms). Let  $G$  be a group and  $M$  be an additive  $G$ -module. Let  $n \geq 0$  be an integer and  $C^n(G, M)$  be the additive group of all maps from  $G^n$  to  $M$ . The elements of  $C^n(G, M)$  are called the *n*-cochains. The coboundary homomorphisms

$$d^n : C^n(G, M) \rightarrow C^{n+1}(G, M)$$

are defined as

$$\begin{aligned} (d^n \varphi)(g_1, \dots, g_{n+1}) = & g_1 \cdot \varphi(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^n (-1)^i \varphi(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ & + (-1)^{n+1} \varphi(g_1, \dots, g_n). \end{aligned}$$

**Lemma 2.2.**  $d^{n+1} \circ d^n = 0$  holds, and  $(C(G, M), d)$  becomes a cochain complex.

**Definition 2.3** (Group cohomology). Define the group of *n*-cocycles as

$$Z^n(G, M) = \text{Ker } (d^n),$$

the group of *n*-coboundaries as

$$\begin{cases} B^0(G, M) = 0, \\ B^n(G, M) = \text{Im } (d^{n-1}) \quad (n \geq 1), \end{cases}$$

and the *n*-th cohomology group as

$$H^n(G, M) = Z^n(G, M) / B^n(G, M), \quad (n \geq 1).$$

**Definition 2.4** (Tate cohomology). Let  $G$  be a finite group and  $M$  be an additive  $G$ -module. Define the trace map  $T_G : M \rightarrow M$  as

$$T_G(m) = \sum_{g \in G} g \cdot m,$$

the groups of 0-cocycles and  $(-1)$ -cocycles as

$$\begin{cases} \hat{Z}^0(G, M) = M^G = H^0(G, M), \\ \hat{Z}^{-1}(G, M) = \text{Ker } (T_G), \end{cases}$$

the groups of 0-coboundaries and  $(-1)$ -coboundaries as

$$\begin{cases} \hat{B}^0(G, M) = \text{Im } (T_G), \\ \hat{B}^{-1}(G, M) = \sum_{g \in G} \text{Im } (g - \text{id}_M), \end{cases}$$

and the *n*-th Tate cohomology group as

$$\hat{H}^n(G, M) = \begin{cases} H^n(G, M) & (n \geq 1), \\ \hat{Z}^0(G, M) / \hat{B}^0(G, M) & (n = 0), \\ \hat{Z}^{-1}(G, M) / \hat{B}^{-1}(G, M) & (n = -1), \\ H_{-n-1}(G, M) & (n \leq -2) \end{cases}$$

where  $H_i$  is the *i*-th homology group.

Assume that  $G$  is a finite group and  $M$  is a  $G$ -lattice, i.e. finitely generated  $\mathbb{Z}[G]$ -module which is  $\mathbb{Z}$ -free as an abelian group. Both  $\hat{Z}^n(G, M)$  and  $\hat{B}^n(G, M)$  are free  $\mathbb{Z}$ -modules of finite rank, and it turns out that the groups  $\hat{H}^n(G, M)$  have exponent dividing  $\#G$  and hence finite for any  $n \in \mathbb{Z}$ . We have that  $\hat{H}^n(G, \mathbb{Z}[G]) = 0$  for  $n = \pm 1$  (see Lemma 2.11) and that  $\hat{H}^0(G, \mathbb{Z}[G]) = 0$ .

We will give some GAP algorithms for computing the Tate cohomology  $\hat{H}^n(G, M_G)$  in Section 8.

In order to construct a flabby resolution of  $M$ , we use the following long exact sequence:

**Lemma 2.5** (A long exact sequence). *If*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

*is an exact sequence of  $G$ -lattices, then there exists a long exact sequence of abelian groups*

$$\cdots \rightarrow \widehat{H}^{k-1}(G, C) \xrightarrow{d^*} \widehat{H}^k(G, A) \xrightarrow{f^*} \widehat{H}^k(G, B) \xrightarrow{g^*} \widehat{H}^k(G, C) \xrightarrow{d^*} \widehat{H}^{k+1}(G, A) \rightarrow \cdots$$

*where  $f^*$  and  $g^*$  are the maps in cohomology induced from the cochain maps  $f$  and  $g$ , and  $d^*$  is the connecting homomorphism obtained by using the snake lemma.*

**Definition 2.6** (Dual  $G$ -lattice). Let  $M$  be a  $G$ -lattice. The  $G$ -lattice  $M^\circ = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  with  $G$ -action

$$(g \cdot f)(m) = f(g^{-1} \cdot m) \quad (f \in M^\circ, m \in M, g \in G)$$

is called the dual  $G$ -lattice of  $M$ .

Note that  $(M^\circ)^\circ \simeq M$  and if  $A(g)$  is the matrix representation of the action of  $g \in G$  on the  $G$ -lattice  $M$  with a fixed  $\mathbb{Z}$ -basis, then the matrix representation of the action of  $g$  on  $M^\circ$  is given by  ${}^t A(g^{-1})$ . In particular, if  $M$  is permutation  $G$ -lattice, then  $M \simeq M^\circ$ .

**Lemma 2.7** (see [Arn84, Theorem 2.2]).  $H^n(G, M) \simeq \widehat{H}^{-n}(G, M^\circ)$ .

**Definition 2.8** (Dual homomorphism). For  $f \in \text{Hom}_{\mathbb{Z}}(M_1, M_2)$ , we define the dual homomorphism  $f^\circ \in \text{Hom}_{\mathbb{Z}}(M_2^\circ, M_1^\circ)$  by

$$f^\circ(l)(m) = l(f(m)) \quad (l \in M_2^\circ, m \in M_1).$$

**Lemma 2.9.** *If  $A \xrightarrow{f} B \xrightarrow{g} C$  is an exact sequence of  $G$ -lattices, then  $C^\circ \xrightarrow{g^\circ} B^\circ \xrightarrow{f^\circ} A^\circ$  is also an exact sequence of  $G$ -lattices.*

We recall some basic facts of the theory of flabby (flasque)  $G$ -lattices (see [CTS77], [Swa83], [Vos98, Chapter 2], [Lor05], [Swa10]).

**Definition 2.10** (Permutation, stably permutation, invertible, flabby and coflabby  $G$ -lattices). Let  $G$  be a finite group and  $M$  be a  $G$ -lattice (i.e. finitely generated  $\mathbb{Z}[G]$ -module which is  $\mathbb{Z}$ -free as an abelian group). A  $G$ -lattice  $M$  is called permutation if  $M$  has a  $\mathbb{Z}$ -basis permuted by  $G$ , i.e.  $M \simeq \bigoplus_{1 \leq i \leq m} \mathbb{Z}[G/H_i]$  for some subgroups  $H_1, \dots, H_m$  of  $G$ .  $M$  is called stably permutation if  $M \oplus P \simeq P'$  for some permutation  $G$ -lattices  $P$  and  $P'$ .  $M$  is called invertible (or permutation projective) if it is a direct summand of a permutation  $G$ -lattice, i.e.  $P \simeq M \oplus M'$  for some permutation  $G$ -lattice  $P$  and a  $G$ -lattice  $M'$ .  $M$  is called flabby (or flasque) if  $\widehat{H}^{-1}(H, M) = 0$  for any subgroup  $H$  of  $G$  where  $\widehat{H}$  is the Tate cohomology. Similarly,  $M$  is called coflabby (or coflasque) if  $H^1(H, M) = 0$  for any subgroup  $H$  of  $G$ .

**Lemma 2.11** ([Len74, Propositions 1.1 and 1.2], see also [Swa83, Section 8]). *Let  $E$  be an invertible  $G$ -lattice. Then*

- (i)  *$E$  is flabby and coflabby.*
- (ii) *If  $C$  is a coflabby  $G$ -lattice, then any short exact sequence  $0 \rightarrow C \rightarrow N \rightarrow E \rightarrow 0$  splits.*

Let  $C(G)$  be the category of all  $G$ -lattices and  $S(G)$  be the category of all permutation  $G$ -lattices. We say that two  $G$ -lattices  $M_1$  and  $M_2$  are similar if there exist permutation  $G$ -lattices  $P_1$  and  $P_2$  such that  $M_1 \oplus P_1 \simeq M_2 \oplus P_2$ . We denote the similarity class of  $M$  by  $[M]$ . The category of similarity classes  $C(G)/S(G)$  becomes a commutative monoid (with respect to the sum  $[M_1] + [M_2] := [M_1 \oplus M_2]$  and the zero  $0 = [P]$  where  $P \in S(G)$ ).

**Theorem 2.12** (Endo and Miyata [EM74, Lemma 1.1], Colliot-Thélène and Sansuc [CTS77, Lemma 3], see also [Swa83, Lemma 8.5], [Lor05, Lemma 2.6.1]). *For any  $G$ -lattice  $M$ , there is a short exact sequence of  $G$ -lattices  $0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$  where  $P$  is permutation and  $F$  is flabby.*

**Definition 2.13** (Flabby resolution). The exact sequence  $0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$  as in Theorem 2.12 is called a flabby resolution of the  $G$ -lattice  $M$ .  $\rho_G(M) = [F] \in C(G)/S(G)$  is called the flabby class of  $M$ , denoted by  $[M]^{fl} = [F]$ . Note that  $[M]^{fl}$  is well-defined: if  $[M] = [M']$ ,  $[M]^{fl} = [F]$  and  $[M']^{fl} = [F']$  then  $F \oplus P_1 \simeq F' \oplus P_2$  for some permutation  $G$ -lattices  $P_1$  and  $P_2$ , and therefore  $[F] = [F']$  (cf. [Swa83, Lemma 8.7]). We say that  $[M]^{fl}$  is invertible if  $[M]^{fl} = [E]$  for some invertible  $G$ -lattice  $E$ .

It is not difficult to see

$$\begin{array}{ccc}
\text{"}M \text{ is permutation"} & \Rightarrow & \text{"}M \text{ is stably permutation"} \Rightarrow \text{"}M \text{ is invertible"} \Rightarrow \text{"}M \text{ is flabby and coflabby"}. \\
\downarrow & & \downarrow \\
[M]^{fl} = 0 & \Rightarrow & [M]^{fl} \text{ is invertible}
\end{array}$$

(see Section 1). Let  $L/k$  be a finite Galois extension with Galois group  $G = \text{Gal}(L/k)$  and  $M$  be a  $G$ -lattice. The flabby class  $[M]^{fl}$  plays crucial role in the rationality problem for  $L(M)^G$ . In particular, by Theorem 1.16 (Endo and Miyata [EM73], Voskresenskii [Vos74] and Saltman [Sal84a]), we have the following fundamentals:

- (i)  $[M]^{fl} = 0$  if and only if  $L(M)^G$  is stably  $k$ -rational;
- (ii)  $[M]^{fl} = [M']^{fl}$  if and only if  $L(M)^G$  and  $L(M')^G$  are stably isomorphic, i.e. there exist algebraically independent elements  $x_1, \dots, x_m$  over  $L(M)^G$  and  $y_1, \dots, y_n$  over  $L(M')^G$  such that  $L(M)^G(x_1, \dots, x_m) \simeq L(M')^G(y_1, \dots, y_n)$ ;
- (iii)  $[M]^{fl}$  is invertible if and only if  $L(M)^G$  is retract  $k$ -rational.

Let  $G \simeq \text{Gal}(L/k)$  be a finite group and  $M \simeq M_1 \oplus M_2$  be a decomposable  $G$ -lattice. Let  $N_i = \{\sigma \in G \mid \sigma(v) = v \text{ for any } v \in M_i\}$  be the kernel of the action of  $G$  on  $M_i$  ( $i = 1, 2$ ). Then  $L(M)^G$  is the function field of an algebraic torus  $T$  and is  $k$ -isomorphic to the free composite of  $L(M_1)^G$  and  $L(M_2)^G$  over  $k$  where  $L(M_i)^G = (L^{N_i})(M_i^{N_i})^{G/N_i}$  is the function field of some torus  $T_i$  ( $i = 1, 2$ ) with  $T = T_1 \times T_2$  and  $M_i$  may be regarded as a  $G/N_i$ -lattice.

**Lemma 2.14.** *Let  $G$  be a finite group and  $M \simeq M_1 \oplus M_2$  be a decomposable  $G$ -lattice with the flabby class  $\rho_G(M) = [M]^{fl}$ . Let  $N_1$  be a normal subgroup of  $G$  which acts on  $M_1$  trivially. The  $G$ -lattice  $M_1$  may be regarded as a  $G/N_1$ -lattice with the flabby class  $\rho_{G/N_1}(M_1)$  as  $G/N_1$ -lattice. Then*

- (i)  $\rho_G(M) = \rho_G(M_1) + \rho_G(M_2)$ .
- (ii)  $\rho_G(M_1) = 0$  if and only if  $\rho_{G/N_1}(M_1) = 0$ .
- (iii)  $\rho_G(M_1)$  is invertible if and only if  $\rho_{G/N_1}(M_1)$  is invertible.

*Proof.* (i) Let  $0 \rightarrow M_i \rightarrow P_i \rightarrow F_i \rightarrow 0$  be flabby resolutions of  $M_i$  as  $G$ -lattices ( $i = 1, 2$ ). Then  $0 \rightarrow M \rightarrow P_1 \oplus P_2 \rightarrow F_1 \oplus F_2 \rightarrow 0$  is a flabby resolution of  $M$ . Hence  $\rho_G(M) = [F_1 \oplus F_2] = \rho_G(M_1) + \rho_G(M_2)$ . (ii), (iii) See [CTS77, Lemme 2] and [Kan09, Lemma 4.1].  $\square$

**Lemma 2.15** (Swan [Swa10, Lemma 3.1]). *Let  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  be a short exact sequence of  $G$ -lattices with  $M_2$  invertible. Then  $\rho_G(M) = \rho_G(M_1) + \rho_G(M_2)$ . In particular, if  $\rho_G(M_1)$  is invertible, then  $-\rho_G(M_1) = \rho_G(\rho_G(M_1))$ .*

*Proof.* See [Swa10, Lemma 3.1].  $\square$

**Lemma 2.16.** *Let  $G \simeq \text{Gal}(L/k)$  be a finite group and  $M \simeq M_1 \oplus M_2$  be a decomposable  $G$ -lattice.*

- (i)  $L(M)^G$  is retract  $k$ -rational if and only if both of  $L(M_i)^G$  ( $i = 1, 2$ ) are retract  $k$ -rational.
- (ii) If  $L(M_1)^G$  and  $L(M_2)^G$  are stably  $k$ -rational, then  $L(M)^G$  is stably  $k$ -rational.
- (iii) When  $\text{rank } M_i \leq 3$  ( $i = 1, 2$ ),  $L(M)^G$  is stably  $k$ -rational if and only if both of  $L(M_i)^G$  ( $i = 1, 2$ ) are stably  $k$ -rational.

*Proof.* See, for example, [HKKi, Theorem 6.5].  $\square$

Let  $H$  be a subgroup of  $G$ . For a  $G$ -lattice  $M$ , it can be regarded as a  $H$ -lattice by restricting the action of  $G$  to  $H$ . We write this  $H$ -lattice as  $M|_H$ .

**Lemma 2.17.** *Let  $G$  be a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  and  $M_G$  be the corresponding  $G$ -lattice as in Definition 1.24. Let  $H$  be a subgroup of  $G$ .*

- (i) If  $\rho_G(M_G) = 0$ , then  $\rho_H(M_H) = 0$ .
- (ii) If  $\rho_G(M_G)$  is invertible, then  $\rho_H(M_H)$  is invertible.

*Proof.* Let  $0 \rightarrow M_G \rightarrow P \rightarrow F \rightarrow 0$  be a flabby resolution of  $M_G$  as a  $G$ -lattice. Then  $0 \rightarrow M_G|_H \rightarrow P|_H \rightarrow F|_H \rightarrow 0$  is a flabby resolution of  $M_G|_H = M_H$  as a  $H$ -lattice, because  $P_H$  is a permutation  $H$ -lattice and  $F|_H$  is a flabby  $H$ -lattice. This shows that  $\rho_H(M_H) = [F|_H]$  as a  $H$ -lattice. If  $G$ -lattice  $F$  is stably permutation (resp. invertible), then  $F|_H$  is stably permutation (resp. invertible) as a  $H$ -lattice.  $\square$



3. CARAT CODE OF THE  $\mathbb{Z}$ -CLASSES IN DIMENSIONS 5 AND 6

In this section, we will explain how to access the GAP code and the CARAT code of a finite subgroup  $G$  of  $\mathrm{GL}(n, \mathbb{Z})$  ( $n \leq 6$ ). We need the GAP ([GAP]) packages CrystCat and CARAT to do the computations below.

The CrystCat package of GAP provides a catalog of  $\mathbb{Q}$ -classes and  $\mathbb{Z}$ -classes (conjugacy classes) of finite subgroups  $G$  of  $\mathrm{GL}(n, \mathbb{Q})$  and  $\mathrm{GL}(n, \mathbb{Z})$  ( $2 \leq n \leq 4$ ). For  $2 \leq n \leq 4$ , the GAP code  $(n, i, j, k)$  of a finite subgroup  $G$  of  $\mathrm{GL}(n, \mathbb{Z})$  means that  $G$  belongs to the  $k$ -th  $\mathbb{Z}$ -class of the  $j$ -th  $\mathbb{Q}$ -class of the  $i$ -th crystal system of dimension  $n$  in GAP (see also [BBNWZ78, Table 1]).

The CARAT<sup>2</sup> ([Carat]) package of GAP provides all conjugacy classes of finite subgroups of  $\mathrm{GL}(n, \mathbb{Q})$  ( $n \leq 6$ ) (see [PS00]). After unpacking the CARAT, we get the  $\mathbb{Q}$ -catalog file `carat-2.1b1/tables/qcatalog.tar.gz`. Unpacking this file, we get lists of  $\mathbb{Q}$ -classes of  $\mathrm{GL}(n, \mathbb{Q})$  ( $n = 1, \dots, 6$ ) in `qcatalog/data1, \dots, qcatalog/data6`. Generators of each group are in individual files under the folders `qcatalog/dim1, \dots, qcatalog/dim6/`.

The second-named author wrote the perl script `cryst1st.pl` to collect these generators into a single file. Files `cryst1.gap, \dots, cryst6.gap` are lists of representatives of  $\mathbb{Q}$ -classes of  $\mathrm{GL}(1, \mathbb{Q}), \dots, \mathrm{GL}(6, \mathbb{Q})$  respectively. These files are available from <http://www.math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/> as `GlnQ.zip`.

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$ . CARAT has a command `ZClassRepsQClass(G)` to compute the complete  $\mathbb{Z}$ -class representatives of the  $\mathbb{Q}$ -class of  $G$ . We split the  $\mathbb{Q}$ -class of  $G$  into  $\mathbb{Z}$ -classes by the command `ZClassRepsQClass(G)`. For the  $l$ -th group  $\tilde{G}$  in the list of  $\mathbb{Z}$ -classes obtained by `ZClassRepsQClass(G)` where  $G$  is the  $m$ -th group ( $\mathbb{Q}$ -class) in `qcatalog/datan`, we say that the CARAT code of  $\tilde{G}$  is  $(n, m, l)$ .

The second-named author wrote a GAP program to determine the  $\mathbb{Q}$ -class and the  $\mathbb{Z}$ -class of a group  $G$ . The files `crystcat.gap` and `caratnumber.gap` contain programs related to the GAP code and the CARAT code respectively. The file `caratnumber.gap` uses other files which are packed in the `crystdat.zip`. This zip file should be packed at the current directory.

All the files above are available from <http://www.math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/>.

`MatGroupZClass(n, i, j, k)` (build-in function of GAP) returns the group  $G \leq \mathrm{GL}(n, \mathbb{Z})$  of the GAP code  $(n, i, j, k)$  when  $2 \leq n \leq 4$ .

`CaratMatGroupZClass(n, i, j)` returns the group  $G \leq \mathrm{GL}(n, \mathbb{Z})$  of the CARAT code  $(n, i, j)$  when  $1 \leq n \leq 6$ .

`CrystCatZClass(G)` returns the GAP code  $(n, i, j, k)$  of  $G \leq \mathrm{GL}(n, \mathbb{Z})$  when  $1 \leq n \leq 4$ .

`CaratZClass(G)` returns the CARAT code  $(n, i, j)$  of  $G \leq \mathrm{GL}(n, \mathbb{Z})$  when  $1 \leq n \leq 6$ .

`NrQClasses(n)` returns the number of  $\mathbb{Q}$ -classes in dimension  $n$  when  $1 \leq n \leq 6$ .

`NrZClasses(n, m)` returns the number of  $\mathbb{Z}$ -classes in the  $m$ -th  $\mathbb{Q}$ -class in dimension  $n$  when  $1 \leq n \leq 6$ .

`CrystCat2Carat(l)` returns the CARAT code of the group  $G$  of the GAP code  $l$ .

`Carat2CrystCat(l)` returns the GAP code of the group  $G$  of the CARAT code  $l$ .

**Example 3.1** (Some programs in `crystcat.gap` and `caratnumber.gap`). We give some examples of the functions in `crystcat.gap` and `caratnumber.gap`. Note that `caratnumber.gap` needs the CARAT package in GAP.

Let  $C_n$  be the cyclic group of order  $n$  and  $J_n = J_{C_n}$  be the Chevalley module of rank  $n - 1$  which is the dual of  $I_n = \mathrm{Ker} \varepsilon$  where  $\varepsilon : \mathbb{Z}[C_n] \rightarrow \mathbb{Z}$  is the augmentation map (see Section 1). Then  $L(J_n)^{C_n}$  is the function field of the norm one torus  $R_{K/k}^{(1)}(G_m)$  where  $K$  is a cyclic Galois extension of  $k$  of degree 5.

```
gap> Read("crystcat.gap");
gap> Read("caratnumber.gap");

gap> List([1..6], n->NrQClasses(n)); # # of Q-classes in dimension n
[ 2, 10, 32, 227, 955, 7103 ]
gap> List([1..6], n->Sum([1..NrQClasses(n)], i->NrZClasses(n, i))); # # of Z-classes
[ 2, 13, 73, 710, 6079, 85308 ]

gap> J5:=Group([ [ 0, 1, 0, 0 ], [ 0, 0, 1, 0 ], [ 0, 0, 0, 1 ], [ -1, -1, -1, -1 ] ]);
gap> CrystCatZClass(J5);
[ 4, 27, 1, 1 ]
gap> G:=MatGroupZClass(4, 27, 1, 1); # G=C5
```

<sup>2</sup>CARAT works only on Linux or Mac OS X, but not on Windows. Because in the old version of CARAT the number of  $\mathbb{Q}$ -classes of  $\mathrm{GL}(6, \mathbb{Q})$  was not correct, the second-named author reported it to CARAT group. This has been fixed in the current version 2.1b1.

```

MatGroupZClass( 4, 27, 1, 1 )
gap> CrystCat2Carat([4,27,1,1]);
[ 4, 227, 1 ]
gap> Carat2CrystCat([4,227,1]);
[ 4, 27, 1, 1 ]
gap> GeneratorsOfGroup(G);
[ [ 0, -1, 1, 0 ], [ 0, -1, 0, 1 ], [ 0, -1, 0, 0 ], [ 1, -1, 0, 0 ] ] ]
gap> P:=RepresentativeAction(GL(4,Integers),J5,G);
[ [ 0, -1, 0, 1 ], [ 0, 1, 0, 0 ], [ 0, 0, -1, 0 ], [ -1, 0, 1, 0 ] ]
gap> J5^P=G; #checking P^-1*J5*P=G
true

gap> J6:=Group([ [ [ 0, 1, 0, 0, 0 ], [ 0, 0, 1, 0, 0 ], [ 0, 0, 0, 1, 0 ],
> [ 0, 0, 0, 0, 1 ], [ -1, -1, -1, -1, -1 ] ] ]]);
<matrix group with 1 generators>
gap> CaratZClass(J6);
[ 5, 461, 4 ]
gap> G:=CaratMatGroupZClass(5,461,4); # G=C6
<matrix group with 1 generators>
gap> GeneratorsOfGroup(G);
[ [ [ -1, 0, 1, 0, 0 ], [ 1, 0, 0, 0, 0 ], [ -1, 0, 0, 0, -1 ],
    [ 1, 1, 0, 0, 0 ], [ 1, 0, 0, 1, 0 ] ] ] ]
gap> P:=RepresentativeAction(GL(5,Integers),J6,G);
[ [ 0, 0, 0, -1, 1 ], [ 0, -1, 0, 1, 0 ], [ 0, 1, 0, 0, 0 ],
  [ 1, 0, 0, 0, 0 ], [ -1, 0, 1, 0, 0 ] ]
gap> J6^P=G; #checking P^-1*J6*P=G
true

```

Some programs related to a flabby resolution are available from  
<http://www.math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/>.

#### 4. KRULL-SCHMIDT THEOREM FAILS FOR DIMENSION 5

Let  $G$  be a finite group and  $\mathbb{Z}[G]$  be the integral group ring of  $G$ .

**Definition 4.1** (Decomposable and reducible  $G$ -lattices). A  $G$ -lattice  $M$  is said to be reducible if there exists a non-trivial  $G$ -invariant subspace of  $M$ . A  $G$ -lattice is said to be irreducible if it is not reducible. A  $G$ -lattice  $M$  is said to be decomposable if there exist non-trivial  $G$ -lattices  $U_1$  and  $U_2$  such that  $M \simeq U_1 \oplus U_2$ . A  $G$ -lattice is said to be indecomposable if it is not decomposable.

If a  $G$ -lattice  $M$  is decomposable, then it is reducible. By Maschke's theorem, the converse holds for  $\mathbb{Q}[G]$ -modules, but not for  $G$ -lattices (i.e. finitely generated  $\mathbb{Z}$ -free  $\mathbb{Z}[G]$ -module).

We say that the direct sum cancellation holds for  $G$ -lattices if  $M_1 \oplus N \simeq M_2 \oplus N$  implies  $M_1 \simeq M_2$  for  $G$ -lattices  $M_1, M_2$  and  $N$ . We say that the Krull-Schmidt theorem holds for  $G$ -lattices if  $M_G \simeq M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $M_i \simeq N_i$  for any  $1 \leq i \leq m$ . Clearly, the Krull-Schmidt theorem implies the direct sum cancellation.

By Krull-Schmidt-Azumaya theorem [Azu50] (see [CR81, Theorem 6.12]), the Krull-Schmidt theorem holds for any  $\mathbb{Z}_p[G]$ -lattice where  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers. By [Jon65, Theorem 2] (see [CR81, Theorem 36.1]), for  $p$ -group  $G$  where  $p$  is odd prime, the Krull-Schmidt theorem holds for  $\mathbb{Z}_{(p)}[G]$ -lattices where  $\mathbb{Z}_{(p)}$  is the localization of  $\mathbb{Z}$  at the prime ideal  $(p)$ . Note that the  $G$ -lattice  $\mathbb{Z}[G/H]$  is indecomposable for any subgroup  $H \leq G$  (see [CR87, Theorem 32.14]).

Endo and Hironaka [EH79, Theorem, page 161] (see [CR87, Theorem 50.29]) showed that if the direct sum cancellation holds for  $G$ -lattices, then  $G$  is abelian, dihedral,  $A_4$ ,  $S_4$  or  $A_5$  via the projective class group (see also [Swa88, Corollary 1.3 and Section 7]). The question whether the Krull-Schmidt theorem holds for  $G$ -lattices is solved except for the case where  $G$  is the dihedral group  $D_8$  of order 16 (see also [Fac03, Theorem 9.1]).

**Theorem 4.2** (Hindman, Klingler and Odenthal [HKO98, Theorem 1.6]). *Let  $G$  be a finite group which is not the dihedral group  $D_8$  of order 16. Then the Krull-Schmidt theorem holds for  $G$ -lattices if and only if one of the*

following conditions holds: (i)  $G = C_p$  for prime  $p \leq 19$ ; (ii)  $G = C_n$  for  $n = 1, 4, 8$  or  $9$ ; (iii)  $G = C_2 \times C_2$ ; (iv)  $G$  is the dihedral group  $D_4$  of order 8.

Two  $G$ -lattices  $M, N$  are placed in the same genus ( $M \approx N$ ) if  $M_{(p)} \simeq N_{(p)}$  for any prime ideal  $(p)$ . We say that the generalized Krull-Schmidt theorem holds for  $G$ -lattices if  $M_G \simeq M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $M_i \approx N_i$  for any  $1 \leq i \leq m$ . Clearly, the Krull-Schmidt theorem implies the generalized Krull-Schmidt theorem. The following theorem was pointed out to the authors by S. Endo:

**Theorem 4.3** (Dress [Dre70, Section 1]). *Let  $p$  be a prime number. The following conditions are equivalent:*

- (i)  $G$  is a  $p$ -group;
- (ii) the generalized Krull-Schmidt theorem holds for invertible  $G$ -lattices.

*Proof.* (This proof is due to S. Endo [End12].) (i)  $\Rightarrow$  (ii). If  $p$  is odd prime, then the Krull-Schmidt theorem holds for  $\mathbb{Z}_{(p)}[G]$ -lattices. Assume that  $G$  is a 2-group and  $M$  is an invertible  $G$ -lattice. Since the Krull-Schmidt theorem holds for  $\mathbb{Z}_2[G]$ -lattices,  $M_2 \simeq \bigoplus \mathbb{Z}_2[G/H_i]$  where  $M_2$  is the 2-adic completion of  $M$  and  $H_i \leq G$ . It follows from Maranda's theorem [CR81, Theorem 30.14] (see also [CR81, Proposition 30.17]) that  $M_{(2)} \simeq \bigoplus \mathbb{Z}_{(2)}[G/H_i]$  where  $M_{(2)}$  is the localization of  $M$  at prime  $(2)$ . The generalized Krull-Schmidt theorem holds for invertible  $G$ -lattices because  $\mathbb{Z}_{(p)}[G/H]$  is indecomposable for any prime  $p$  and any subgroup  $H \leq G$ .

(ii)  $\Rightarrow$  (i). Assume that  $G$  is not a  $p$ -group and  $H \leq G$  is of order  $p^l q^m$  where  $p$  and  $q$  are different primes and  $l, m \geq 1$ . Let  $Sy_p(H)$  be a  $p$ -Sylow subgroup of  $H$ . Then by [Dre70, Section 1] there exists  $G$ -lattice  $M$  such that  $\mathbb{Z}[H/Sy_p(H)] \oplus \mathbb{Z}[H/Sy_q(H)] \simeq M \oplus \mathbb{Z}$ . By taking the tensor product  $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]}$  of both sides, we get  $\mathbb{Z}[G/Sy_p(H)] \oplus \mathbb{Z}[G/Sy_q(H)] \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \oplus \mathbb{Z}[G/H]$  as  $G$ -lattices. This contradicts that the generalized Krull-Schmidt theorem holds.  $\square$

A  $G$ -set is a finite set with left  $G$ -action. The disjoint union  $X \coprod X'$  of  $G$ -sets  $X$  and  $X'$  is also  $G$ -set. Two  $G$ -sets are isomorphic if there exists a bijection between them which preserves the action of  $G$ . A  $G$ -set  $X$  may be written uniquely up to isomorphism as  $X \simeq \coprod_H a_H(X)G/H$  where  $H$  runs through a set of representatives of conjugacy classes of subgroups of  $G$  (see [CR87, Chapter 11], [Ben91, Chapter 5], [GW93], [Bou00] for related materials, e.g. Burnside ring). For  $G$ -sets  $X$  and  $X'$ , the direct sum of permutation  $G$ -lattices  $\mathbb{Z}[X]$  and  $\mathbb{Z}[X']$  is also permutation:  $\mathbb{Z}[X] \oplus \mathbb{Z}[X'] \simeq \mathbb{Z}[X \coprod X']$ . A finite group  $G$  is called cyclic mod  $p$  (or  $p$ -hypoelementary) if the quotient group  $G/O_p(G)$  is cyclic where  $O_p(G)$  is the largest normal  $p$ -subgroup of  $G$ .

**Theorem 4.4** (Dress [Dre73, Proposition 9.6]). *Let  $G$  be a finite group. The following conditions are equivalent:*

- (i)  $G$  is cyclic mod  $p$  for some  $p$ ;
- (ii) for any two  $G$ -sets  $S, T$ ,  $\mathbb{Z}[S] \simeq \mathbb{Z}[T]$  if and only if  $S \simeq T$ .

*In particular, the Krull-Schmidt theorem holds for permutation  $G$ -lattices if and only if  $G$  is cyclic mod  $p$  for some  $p$ .*

Using the algorithms in Sections 5 and 6, we will show the following proposition by constructing explicit isomorphism in Section 6 (see Example 6.8). We remark that the dihedral group  $D_6$  of order 12 is the smallest group which is not cyclic mod  $p$  for any  $p$ .

**Proposition** (Proposition 6.7). *Let  $D_6$  be the dihedral group of order 12 and  $\{1\}, C_2^{(1)}, C_2^{(2)}, C_2^{(3)}, C_3, C_2^2, C_6, S_3^{(1)}, S_3^{(2)}$  and  $D_6$  be the conjugacy classes of subgroups of  $D_6$ . Then the following isomorphism of permutation  $D_6$ -lattices holds:*

$$\begin{aligned} & \mathbb{Z}[D_6] \oplus 2\mathbb{Z}[D_6/C_2^2] \oplus \mathbb{Z}[D_6/C_6] \oplus \mathbb{Z}[D_6/S_3^{(1)}] \oplus \mathbb{Z}[D_6/S_3^{(2)}] \\ & \simeq \mathbb{Z}[D_6/C_2^{(1)}] \oplus \mathbb{Z}[D_6/C_2^{(2)}] \oplus \mathbb{Z}[D_6/C_2^{(3)}] \oplus \mathbb{Z}[D_6/C_3] \oplus 2\mathbb{Z}. \end{aligned}$$

**Definition 4.5** (Decomposition type). Let  $G$  be a finite group and  $M$  be a  $G$ -lattice. When  $M$  decomposes into indecomposable  $G$ -lattices  $M \simeq U_1 \oplus \cdots \oplus U_r$  of rank  $n_1, \dots, n_r$ , we say that a decomposition type  $\text{DT}(M)$  of  $M$  is  $(n_1, \dots, n_r)$ . (This may not be unique.)

Let  $G$  be a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  and  $M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. The number of  $G$ -lattices  $M_G$  for a given decomposition type  $\text{DT}(M_G)$  is as follows (see Example 4.9 below):

$\text{DT}(M_G)$	(1)	Total	$\text{DT}(M_G)$	(1, 1)	(2)	Total	$\text{DT}(M_G)$	(1, 1, 1)	(2, 1)	(3)	Total
$\#M_G$	2	2	$\#M_G$	4	9	13	$\#M_G$	8	31	34	73

DT( $M_G$ )	(1, 1, 1, 1)	(2, 1, 1)	(2, 2)	(3, 1)	(4)	Total
# $M_G$	16	96	175	128	295	710

DT( $M_G$ )	(1 <sup>5</sup> )	(2, 1 <sup>3</sup> )	(2 <sup>2</sup> , 1)	(3, 1 <sup>2</sup> )	(3, 2)	(4, 1)	(5)	Total
# $M_G$	32	280	1004	442	<b>1480</b>	<b>1400</b>	1452	<b>6090</b> (6079)

DT( $M_G$ )	(1 <sup>6</sup> )	(2, 1 <sup>4</sup> )	(2 <sup>2</sup> , 1 <sup>2</sup> )	(2 <sup>3</sup> )	(3, 1 <sup>3</sup> )	(3, 2, 1)	(3 <sup>2</sup> )	(4, 1 <sup>2</sup> )	(4, 2)	(5, 1)	(6)	Total
# $M_G$	68	824	4862	6878	1466	<b>10662</b>	<b>4235</b>	<b>5944</b>	21573	<b>9931</b>	18996	<b>85439</b> (85308)

For  $n \leq 4$ , we see that the Krull-Schmidt theorem holds for  $G$ -lattices. However, it turns out that the Krull-Schmidt theorem fails for  $n = 5$ . We split the Krull-Schmidt theorem into the following two parts:

(KS1) If  $M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $\text{rank } M_i = \text{rank } N_i$  for any  $1 \leq i \leq m$ ;

(KS2) If  $M_1 \oplus \cdots \oplus M_m \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_i$  with  $\text{rank } M_i = \text{rank } N_i$  for any  $1 \leq i \leq m$ , then after a suitable renumbering of the  $N_i$ ,  $M_i \simeq N_i$  for any  $1 \leq i \leq m$ .

Krull-Schmidt theorem holds if and only if the conditions (KS1) and (KS2) hold.

**Theorem 4.6.** *Let  $G$  be a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24.*

- (i) *When  $n \leq 4$ , the Krull-Schmidt theorem holds, i.e. if  $M_G \simeq M_1 \oplus \cdots \oplus M_l \simeq N_1 \oplus \cdots \oplus N_m$  for indecomposable  $G$ -lattices  $M_i$  and  $N_j$ , then  $l = m$  and, after a suitable renumbering of the  $N_j$ ,  $M_i \simeq N_i$  for any  $1 \leq i \leq m$ .*
- (ii) *When  $n = 5$ , (KS2) holds, and the Krull-Schmidt theorem fails if and only if (KS1) fails if and only if the CARAT code of  $G$  is one of the 11 triples*

$$(5, 188, 4), (5, 189, 4), (5, 190, 6), (5, 191, 6), (5, 192, 6), (5, 193, 4), (5, 205, 6), (5, 218, 8), (5, 219, 8), (5, 220, 4), (5, 221, 4).$$

*For the exceptional 11 cases, the decomposition types of  $M_G$  are (3, 2) and (4, 1) and  $G$  is a subgroup of the group  $C_2 \times D_6$  of the CARAT code (5, 205, 6).*

- (iii) *When  $n = 6$ , (KS1) fails if and only if the CARAT code of  $G$  is one of the 131 triples*

$$\begin{aligned} &(6, 2013, 8), (6, 2018, 4), (6, 2023, 6), (6, 2024, 6), (6, 2025, 6), (6, 2026, 6), (6, 2033, 6), (6, 2042, 8), (6, 2043, 8), (6, 2044, 4), \\ &(6, 2045, 4), (6, 2048, 5), (6, 2049, 8), (6, 2050, 8), (6, 2051, 8), (6, 2052, 8), (6, 2058, 5), (6, 2059, 5), (6, 2067, 5), (6, 2068, 5), \\ &(6, 2069, 5), (6, 2069, 11), (6, 2070, 9), (6, 2071, 9), (6, 2072, 10), (6, 2072, 11), (6, 2076, 24), (6, 2076, 25), (6, 2077, 24), (6, 2077, 25), \\ &(6, 2078, 24), (6, 2078, 25), (6, 2079, 24), (6, 2079, 25), (6, 2087, 15), (6, 2088, 15), (6, 2089, 17), (6, 2089, 18), (6, 2094, 9), (6, 2102, 24), \\ &(6, 2102, 25), (6, 2105, 9), (6, 2106, 9), (6, 2107, 10), (6, 2107, 11), (6, 2108, 15), (6, 2109, 15), (6, 2110, 17), (6, 2110, 18), (6, 2111, 15), \\ &(6, 2139, 9), \\ &(6, 40, 4), (6, 41, 4), (6, 44, 6), (6, 45, 6), (6, 47, 4), (6, 53, 4), (6, 54, 4), (6, 54, 8), (6, 55, 4), (6, 63, 4), \\ &(6, 64, 6), (6, 65, 4), (6, 66, 6), (6, 67, 6), (6, 75, 4), (6, 75, 8), (6, 76, 8), (6, 76, 12), (6, 77, 8), (6, 77, 12), \\ &(6, 78, 4), (6, 78, 8), (6, 79, 6), (6, 80, 4), (6, 81, 8), (6, 81, 12), (6, 90, 4), (6, 99, 4), (6, 108, 4), (6, 108, 8), \\ &(6, 109, 8), (6, 109, 12), (6, 110, 4), (6, 111, 6), (6, 112, 8), (6, 112, 12), (6, 113, 4), (6, 114, 6), (6, 115, 6), (6, 145, 4), \\ &(6, 2070, 10), (6, 2070, 11), (6, 2071, 10), (6, 2071, 11), (6, 2072, 12), (6, 2072, 13), (6, 2076, 26), (6, 2076, 27), (6, 2077, 26), (6, 2077, 27), \\ &(6, 2078, 26), (6, 2078, 27), (6, 2079, 26), (6, 2079, 27), (6, 2087, 16), (6, 2087, 17), (6, 2088, 16), (6, 2088, 17), (6, 2089, 19), (6, 2089, 20), \\ &(6, 2094, 10), (6, 2094, 11), (6, 2102, 26), (6, 2102, 27), (6, 2105, 10), (6, 2105, 11), (6, 2106, 10), (6, 2106, 11), (6, 2107, 12), (6, 2107, 13), \\ &(6, 2108, 16), (6, 2108, 17), (6, 2109, 16), (6, 2109, 17), (6, 2110, 19), (6, 2110, 20), (6, 2111, 16), (6, 2111, 17), (6, 2139, 10), (6, 2139, 11). \end{aligned}$$

*For the former 51 cases (resp. the latter 80 cases), the decomposition types of  $M_G$  are (3, 2, 1) and (4, 1, 1) (resp. (3, 3) and (5, 1)) and  $G$  is a subgroup of the group  $C_2^2 \times D_6$  of the CARAT code (6, 2139, 9) (resp.  $D_6 \times D_4$  of the CARAT code (6, 145, 4)).*

- (iv) *When  $n = 6$ , (KS2) fails if and only if the CARAT code of  $G$  is one of the 18 triples*

$$\begin{aligned} &(6, 2072, 14), (6, 2076, 28), (6, 2077, 28), (6, 2078, 28), (6, 2079, 28), (6, 2089, 21), (6, 2102, 28), (6, 2107, 14), (6, 2110, 21), (6, 2295, 2), \\ &(6, 3045, 3), (6, 3046, 3), (6, 3047, 3), (6, 3052, 5), (6, 3053, 5), (6, 3054, 3), (6, 3061, 5), (6, 3066, 3). \end{aligned}$$

*For the former 10 cases, the decomposition type of  $M_G$  is (4, 2) and  $G$  is the group  $D_6$  of the CARAT code (6, 2295, 2) or a subgroup of the 3 groups  $C_2 \times D_6$  of the CARAT codes (6, 2102, 28), (6, 2107, 14) and (6, 2110, 21). For the latter 8 cases, the decomposition type of  $M_G$  is (5, 1) and  $G$  is a subgroup of the group  $C_2 \times S_5$  of the CARAT code (6, 3054, 3).*

**4.0. Classification of indecomposable maximal finite groups of dimension  $n \leq 6$ .** Let  $G \leq \text{GL}(n, \mathbb{Z})$  be a finite matrix group.  $G$  is called reducible (resp. irreducible, decomposable, indecomposable) if  $M_G$  is reducible (resp. irreducible, decomposable, indecomposable) where  $M_G$  is the corresponding  $G$ -lattice as in Definition 1.24. Let  $\text{Imf}(n, i, j) \leq \text{GL}(n, \mathbb{Z})$  be the  $j$ -th  $\mathbb{Z}$ -class of the  $i$ -th  $\mathbb{Q}$ -class of the irreducible maximal finite group of dimension  $n$  which corresponds to the build-in function `ImfMatrixGroup(n,i,j)` of GAP. For  $n \leq 10$  and  $n = p \leq 23$ ; prime, the irreducible maximal finite groups  $\text{Imf}(n, i, j)$  is determined by Plesken and Pohst [PP77] ( $n \leq 7$ ), [PP80] ( $n = 8, 9$ ), Plesken [Ple85] ( $n = p \leq 23$ ; prime) and Souvignier [Sou94] ( $n = 10$ ).

For  $n = 2$ , there exist exactly 2 irreducible maximal finite groups  $\text{Imf}(2, 1, 1) \simeq D_4$  and  $\text{Imf}(2, 2, 1) \simeq D_6$  of order 8 and 12 of the GAP codes (2, 3, 2, 1) and (2, 4, 4, 1).

For  $n = 3$ , there exist exactly 3 irreducible maximal finite groups  $\text{Imf}(3, 1, 1) \simeq \text{Imf}(3, 1, 2) \simeq \text{Imf}(3, 1, 3) \simeq C_2 \times S_4$  of order 48 of the GAP codes (3, 7, 5, 1), (3, 7, 5, 2) and (3, 7, 5, 3).

For  $n = 4$ , there exist exactly 6 irreducible maximal finite groups  $\text{Imf}(4, 1, 1)$ ,  $\text{Imf}(4, 2, 1) \simeq D_6^2 \rtimes C_2$ ,  $\text{Imf}(4, 3, 1) \simeq \text{Imf}(4, 3, 2) \simeq C_2 \times S_5$ ,  $\text{Imf}(4, 4, 1) \simeq C_2^4 \rtimes S_4$  and  $\text{Imf}(4, 5, 1) \simeq C_2 \times (S_3^2 \rtimes C_2)$  of order 1152, 288, 240, 240, 384 and 144 of the GAP codes (4, 33, 16, 1), (4, 30, 13, 1), (4, 31, 7, 1), (4, 31, 7, 2), (4, 32, 21, 1) and (4, 29, 9, 1) respectively.

For  $n = 5$ , there exist exactly 7 irreducible maximal finite groups  $\text{Imf}(5, 1, 1) \simeq \text{Imf}(5, 1, 2) \simeq \text{Imf}(5, 1, 3) \simeq C_2^5 \rtimes S_5$  of order 3840 of the CARAT codes (5, 942, 1), (5, 942, 2), (5, 942, 3) and  $\text{Imf}(5, 2, 1) \simeq \text{Imf}(5, 2, 2) \simeq \text{Imf}(5, 2, 3) \simeq \text{Imf}(5, 2, 4) \simeq C_2 \times S_6$  of order 1440 of the CARAT codes (5, 949, 1), (5, 949, 4), (5, 949, 2), (5, 949, 3).

For  $n = 6$ , there exist exactly 17 irreducible maximal finite groups  $\text{Imf}(6, 1, 1) \simeq \text{Imf}(6, 1, 2) \simeq \text{Imf}(6, 1, 3) \simeq C_2^6 \rtimes S_6$ ,  $\text{Imf}(6, 2, 1) \simeq D_6^3 \rtimes S_3$ ,  $\text{Imf}(6, 3, 1) \simeq \text{Imf}(6, 3, 2)$ ,  $\text{Imf}(6, 4, 1) \simeq \text{Imf}(6, 4, 2) \simeq C_2 \times S_7$ ,  $\text{Imf}(6, 5, 1) \simeq C_2 \times \text{PGL}(2, \mathbb{F}_7)$ ,  $\text{Imf}(6, 6, 1) \simeq \text{Imf}(6, 6, 2) \simeq \text{Imf}(6, 6, 3) \simeq C_2 \times S_5$ ,  $\text{Imf}(6, 7, 1) \simeq \text{Imf}(6, 7, 2) \simeq (C_2 \times S_4)^2 \rtimes C_2$ ,  $\text{Imf}(6, 8, 1) \simeq (C_2^5 \rtimes A_6) \rtimes C_2$ ,  $\text{Imf}(6, 9, 1) \simeq \text{Imf}(6, 9, 2) \simeq D_6 \times S_4$  of order 46080, 46080, 46080, 10368, 103680, 103680, 10080, 10080, 672, 240, 240, 240, 4608, 4608, 23040, 288 and 288 of the CARAT codes (6, 2773, 1), (6, 2773, 3), (6, 2773, 2), (6, 2803, 1), (6, 2804, 2), (6, 2804, 1), (6, 2932, 1), (6, 2932, 2), (6, 2945, 1), (6, 2952, 1), (6, 2952, 3), (6, 2952, 2), (6, 2772, 2), (6, 2772, 5), (6, 2750, 4), (6, 2866, 2) and (6, 2866, 3) respectively.

Let  $\text{Indmf}(n, i, j) \leq \text{GL}(n, \mathbb{Z})$  be the  $j$ -th  $\mathbb{Z}$ -class of the  $i$ -th  $\mathbb{Q}$ -class of the indecomposable maximal finite group of dimension  $n$ . We see that all the groups  $\text{Indmf}(n, i, j)$  coincide with  $\text{Imf}(n, i, j)$  for  $n \leq 5$ . However, this is not true for  $n = 6$ . This phenomenon is suggested by [Ple78, Section V] (see also [PH84, Section V]). Indeed, it turns out that we need the one additional group  $\text{Indmf}(6, 10, 1) \simeq (C_2 \times S_4)^2$  of order 2304 of the CARAT code (6, 5517, 4) in order to get all the indecomposable maximal finite groups. Namely, there exist exactly 18 indecomposable maximal finite groups of dimension 6.

We will check this in the next subsection (see Example 4.9 in Subsection 4.1 and Example 8.2). The algorithms given in this section are available from <http://math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/> as `KS.gap`.

`IndmfMatrixGroup(n,i,j)` returns  $\text{Indmf}(n, i, j)$  of dimension  $n$  (this works only for  $n \leq 6$ ).

`IndmfNumberQClasses(n)` returns the number of  $\mathbb{Q}$ -classes of all the indecomposable maximal finite groups of dimension  $n$  (this works only for  $n \leq 6$ ).

`IndmfNumberZClasses(n,i)` returns the number of  $\mathbb{Z}$ -classes in the  $i$ -th  $\mathbb{Q}$ -class of the indecomposable maximal finite groups  $\text{Imf}(n, i, j)$  of dimension  $n$  (this works only for  $n \leq 6$ ).

`AllImfMatrixGroups(n)` returns all the irreducible maximal finite groups of dimension  $n$ .

`AllIndmfMatrixGroups(n)` returns all the indecomposable maximal finite groups of dimension  $n$  (this works only for  $n \leq 6$ ).

**Algorithm Indmf** (Constructing all the indecomposable maximal finite groups ( $\text{Indmf}$ ) of dimension  $n \leq 6$ ).  
(The following algorithm needs the CARAT package of GAP and `caratnumber.gap`).

```
IndmfMatrixGroup:= function(d,q,z)
  local ans;
  if d=6 and q=10 and z=1 then
    ans:=CaratMatGroupZClass(6,5517,4);
    SetName(ans,"IndmfMatrixGroup(6,10,1)");
    return ans;
  else
    return ImfMatrixGroup(d,q,z);
  fi;
end;
```

```

end;

IndmfNumberQClasses:= function(d)
  if d=6 then
    return 10;
  else
    return ImfNumberQClasses(d);
  fi;
end;

IndmfNumberZClasses:= function(d,q)
  if d=6 and q=10 then
    return 1;
  else
    return ImfNumberZClasses(d,q);
  fi;
end;

AllImfMatrixGroups:= function(n)
  local l;
  l:=List([1..ImfNumberQClasses(n)],
    x->List([1..ImfNumberZClasses(n,x)],y->ImfMatrixGroup(n,x,y)));
  return Concatenation(l);
end;

AllIndmfMatrixGroups:= function(n)
  local l;
  l:=List([1..ImfNumberQClasses(n)],
    x->List([1..ImfNumberZClasses(n,x)],y->ImfMatrixGroup(n,x,y)));
  l:=Concatenation(l);
  if n=6 then
    l[18]:=IndmfMatrixGroup(6,10,1);
  fi;
  return l;
end;

```

**Example 4.7** (All the indecomposable maximal finite groups of dimension  $n \leq 6$ ).

```

gap> Imf2:=AllImfMatrixGroups(2); # Imf2=Indmf2: 2 groups
[ ImfMatrixGroup(2,1,1), ImfMatrixGroup(2,2,1) ]
gap> List(Imf2,CrystCatZClass);
[ [ 2, 3, 2, 1 ], [ 2, 4, 4, 1 ] ]
gap> List(Imf2,Size);
[ 8, 12 ]
gap> List([1..ImfNumberQClasses(2)],
> x->List([1..ImfNumberZClasses(2,x)],y->ImfInvariants(2,x,y).isomorphismType));
[ [ "C2 wr C2 = D8" ], [ "C2 x S3 = C2 x W(A2) = D12" ] ]

gap> Imf3:=AllImfMatrixGroups(3); # Imf3=Indmf3: 3 groups
[ ImfMatrixGroup(3,1,1), ImfMatrixGroup(3,1,2), ImfMatrixGroup(3,1,3) ]
gap> List(Imf3,CrystCatZClass);
[ [ 3, 7, 5, 1 ], [ 3, 7, 5, 2 ], [ 3, 7, 5, 3 ] ]
gap> List(Imf3,Size);
[ 48, 48, 48 ]
gap> List([1..ImfNumberQClasses(3)],

```

```

> x->List([1..ImfNumberZClasses(3,x)],y->ImfInvariants(3,x,y).isomorphismType));
[ [ "C2 wr S3 = C2 x S4 = W(B3)", "C2 wr S3 = C2 x S4 = C2 x W(A3)",
    "C2 wr S3 = C2 x S4 = C2 x W(A3)" ] ]

gap> Imf4:=AllImfMatrixGroups(4); # Imf4=Indmf4: 6 groups
[ ImfMatrixGroup(4,1,1), ImfMatrixGroup(4,2,1), ImfMatrixGroup(4,3,1),
  ImfMatrixGroup(4,3,2), ImfMatrixGroup(4,4,1), ImfMatrixGroup(4,5,1) ]
gap> List(Imf4,CrystCatZClass);
[ [ 4, 33, 16, 1 ], [ 4, 30, 13, 1 ], [ 4, 31, 7, 1 ], [ 4, 31, 7, 2 ],
  [ 4, 32, 21, 1 ], [ 4, 29, 9, 1 ] ]
gap> List(Imf4,Size);
[ 1152, 288, 240, 240, 384, 144 ]
gap> List([1..ImfNumberQClasses(4)],
> x->List([1..ImfNumberZClasses(4,x)],y->ImfInvariants(4,x,y).isomorphismType));
[ [ "W(F4)" ], [ "D12 wr C2 = (C2 x W(A2)) wr C2" ],
  [ "C2 x S5 = C2 x W(A4)", "C2 x S5 = C2 x W(A4)" ], [ "C2 wr S4 = W(B4)" ],
  [ "(D12 Y D12):C2" ] ]

gap> Imf5:=AllImfMatrixGroups(5); # Imf5=Indmf5: 7 groups
[ ImfMatrixGroup(5,1,1), ImfMatrixGroup(5,1,2), ImfMatrixGroup(5,1,3), ImfMatrixGroup(5,2,1),
  ImfMatrixGroup(5,2,2), ImfMatrixGroup(5,2,3), ImfMatrixGroup(5,2,4) ]
gap> List(Imf5,CaratZClass);
[ [ 5, 942, 1 ], [ 5, 942, 2 ], [ 5, 942, 3 ], [ 5, 949, 1 ],
  [ 5, 949, 4 ], [ 5, 949, 2 ], [ 5, 949, 3 ] ]
gap> List(Imf5,Size);
[ 3840, 3840, 3840, 1440, 1440, 1440, 1440 ]
gap> List([1..ImfNumberQClasses(5)],
> x->List([1..ImfNumberZClasses(5,x)],y->ImfInvariants(5,x,y).isomorphismType));
[ [ "C2 wr S5 = W(B5)", "C2 wr S5 = C2 x W(D5)", "C2 wr S5 = C2 x W(D5)" ],
  [ "C2 x S6", "C2 x S6", "C2 x S6", "C2 x S6" ] ]

gap> Imf6:=AllImfMatrixGroups(6); # Imf6: 17 groups
[ ImfMatrixGroup(6,1,1), ImfMatrixGroup(6,1,2), ImfMatrixGroup(6,1,3), ImfMatrixGroup(6,2,1),
  ImfMatrixGroup(6,3,1), ImfMatrixGroup(6,3,2), ImfMatrixGroup(6,4,1), ImfMatrixGroup(6,4,2),
  ImfMatrixGroup(6,5,1), ImfMatrixGroup(6,6,1), ImfMatrixGroup(6,6,2), ImfMatrixGroup(6,6,3),
  ImfMatrixGroup(6,7,1), ImfMatrixGroup(6,7,2), ImfMatrixGroup(6,8,1), ImfMatrixGroup(6,9,1),
  ImfMatrixGroup(6,9,2) ]
gap> List(Imf6,CaratZClass);
[ [ 6, 2773, 1 ], [ 6, 2773, 3 ], [ 6, 2773, 2 ], [ 6, 2803, 1 ], [ 6, 2804, 2 ],
  [ 6, 2804, 1 ], [ 6, 2932, 1 ], [ 6, 2932, 2 ], [ 6, 2945, 1 ], [ 6, 2952, 1 ],
  [ 6, 2952, 3 ], [ 6, 2952, 2 ], [ 6, 2772, 2 ], [ 6, 2772, 5 ], [ 6, 2750, 4 ],
  [ 6, 2866, 2 ], [ 6, 2866, 3 ] ]
gap> List(Imf6,Size);
[ 46080, 46080, 46080, 10368, 103680, 103680, 10080, 10080, 672, 240,
  240, 240, 4608, 4608, 23040, 288, 288 ]
gap> List([1..ImfNumberQClasses(6)],
> x->List([1..ImfNumberZClasses(6,x)],y->ImfInvariants(6,x,y).isomorphismType));
[ [ "C2 wr S6 = W(B6)", "C2 wr S6 = C2 x W(D6)", "C2 wr S6 = C2 x W(D6)" ],
  [ "(C2 x S3) wr S3 = (C2 x W(A2)) wr S3 = D12 wr S3" ], [ "C2 x W(E6)", "C2 x W(E6)" ],
  [ "C2 x S7 = C2 x W(A6)", "C2 x S7 = C2 x W(A6)" ], [ "C2 x PGL(2,7)" ],
  [ "C2 x S5", "C2 x S5", "C2 x S5" ],
  [ "(C2 x S4) wr C2 = (C2 x W(A3)) wr C2", "(C2 x S4) wr C2 = (C2 x W(A3)) wr C2" ],
  [ "subgroup of index 2 of C2 wr S6" ],
  [ "C2 x S3 x S4 = D12 x S4 = C2 x W(A2) x W(A3)",
    "C2 x S3 x S4 = D12 x S4 = C2 x W(A2) x W(A3)" ] ]

```

```

gap> Indmf2:=AllIndmfMatrixGroups(2);;
gap> Indmf3:=AllIndmfMatrixGroups(3);;
gap> Indmf4:=AllIndmfMatrixGroups(4);;
gap> Indmf5:=AllIndmfMatrixGroups(5);;
gap> Indmf6:=AllIndmfMatrixGroups(6);;
gap> [Imf2=Indmf2,Imf3=Indmf3,Imf4=Indmf4,Imf5=Indmf5,Imf6=Indmf6];
[ true, true, true, true, false ]

gap> Indmf6; # Indmf6: 18 (=17+1) groups
[ ImfMatrixGroup(6,1,1), ImfMatrixGroup(6,1,2), ImfMatrixGroup(6,1,3),
  ImfMatrixGroup(6,2,1), ImfMatrixGroup(6,3,1), ImfMatrixGroup(6,3,2),
  ImfMatrixGroup(6,4,1), ImfMatrixGroup(6,4,2), ImfMatrixGroup(6,5,1),
  ImfMatrixGroup(6,6,1), ImfMatrixGroup(6,6,2), ImfMatrixGroup(6,6,3),
  ImfMatrixGroup(6,7,1), ImfMatrixGroup(6,7,2), ImfMatrixGroup(6,8,1),
  ImfMatrixGroup(6,9,1), ImfMatrixGroup(6,9,2), IndmfMatrixGroup(6,10,1) ]
gap> CaratZClass(IndmfMatrixGroup(6,10,1));
[ 6, 5517, 4 ]
gap> Size(IndmfMatrixGroup(6,10,1));
2304
gap> StructureDescription(IndmfMatrixGroup(6,10,1));
"C2 x C2 x S4 x S4"

```

**4.1. Krull-Schmidt theorem (1).** We will determine all the possible decompositions of  $M_G$  into indecomposable ones for all finite subgroups  $G$  of  $\mathrm{GL}(n, \mathbb{Z})$  with  $n \leq 6$  (see Examples 4.9 and Examples 4.13 below). Note that if a  $G$ -lattice  $M$  splits into indecomposable  $G$ -lattices  $M_1$  and  $M_2$  of rank  $i$  and  $j$ , then  $G$  is a subgroup of  $G_1 \times G_2$  where  $G_1$  and  $G_2$  are some indecomposable maximal finite subgroups of  $\mathrm{GL}(i, \mathbb{Z})$  and  $\mathrm{GL}(j, \mathbb{Z})$  respectively.

`LatticeDecompositions(n)` returns the list  $\mathcal{L} = \{l_1, \dots, l_s\}$  of the lists  $l_i$  of the GAP codes whose  $i$ -th list  $l_i$  contains all the GAP codes of the groups  $G$  whose corresponding  $G$ -lattice  $M_G$  is decomposable into the indecomposable  $G$ -lattices  $M \simeq M_1 \oplus \dots \oplus M_m$  with rank  $M_j = r_j$  where  $(r_1, \dots, r_m)$  corresponds to the  $i$ -th partitions `Partitions(n)[i]` of the integer  $2 \leq n \leq 4$ .

`LatticeDecompositions(n:Carat)` returns the same as `LatticeDecompositions(n)` but using the CARAT code instead of the GAP code. This algorithm is valid for  $1 \leq n \leq 6$ .

`LatticeDecompositions(n:Carat,FromPerm)` returns the same as `LatticeDecompositions(n:Carat)` but using `ConjugacyClassesSubgroupsFromPerm(G)` instead of `ConjugacyClassesSubgroups2(G)` (see below).

In order to construct conjugacy classes of subgroups of a group  $G$ , we use the following GAP function `ConjugacyClassesSubgroups2(G)` because the ordering of the conjugacy classes of subgroups of  $G$  by the built-in function `ConjugacyClassesSubgroups(G)` is not fixed for some groups.

```

ConjugacyClassesSubgroups2:= function(g)
  Reset(GlobalMersenneTwister);
  Reset(GlobalRandomSource);
  return ConjugacyClassesSubgroups(g);
end;

```

If a group  $G$  is too big, `ConjugacyClassesSubgroups2(G)` may not work well. For example, we should use `ConjugacyClassesSubgroupsFromPerm(G)` instead of `ConjugacyClassesSubgroups2(G)` for the 2nd  $\mathbb{Z}$ -class of the 1st  $\mathbb{Q}$ -class of the irreducible maximal finite group  $\mathrm{Imf}(6, 1, 2) \simeq C_2^6 \rtimes S_6$  of dimension 6 of order 46080.



```

ConjugacyClassesSubgroupsFromPerm:= function(g)
  local iso,h,i;
  Reset(GlobalMersenneTwister);
  Reset(GlobalRandomSource);
  iso:=IsomorphismPermGroup(g);
  h:=ConjugacyClassesSubgroups2(Range(iso));
  h:=List(h,Representative);
  h:=List(h,x->PreImage(iso,x));
  return h;
end;

```

**Algorithm KS1** (All the decomposable  $G$ -lattices  $M_G$  of rank  $n$ ).  
*(The following algorithm needs the CARAT package of GAP and caratnumber.gap).*

```

DirectSumMatrixGroup:= function(l)
  local gg,gg1;
  gg:=List(l,GeneratorsOfGroup);
  if Length(Set(gg,Length))>1 then
    return fail;
  else
    gg1:=List([1..Length(gg[1])],x->DirectSumMat(List(gg,y->y[x])));
  fi;
  return Group(gg1,DirectSumMat(List(l,Identity)));
end;

```

```

DirectProductMatrixGroup:= function(l)
  local gg,gg1,o,o1,i,j,gx;
  gg:=List(l,GeneratorsOfGroup);
  gg1:=[];
  for i in [1..Length(l)] do
    o:=List(l,Identity);
    for j in gg[i] do
      o[i]:=j;
      Add(gg1,DirectSumMat(o));
    od;
  od;
  return Group(gg1,DirectSumMat(List(l,Identity)));
end;

```

```

AllDirectProductIndmfMatrixGroups:= function(l)
  local li;
  li:=List(Collected(l),
    x->UnorderedTuples(AllIndmfMatrixGroups(x[1]),x[2]));
  return List(Cartesian(li),
    x->DirectProductMatrixGroup(Concatenation(x)));
end;

```

```

PartialMatrixGroup:= function(G,l)
  local gg,gp;
  gg:=GeneratorsOfGroup(G);
  gp:=List(gg,x->x[1]{1});
  return Group(gp,IdentityMat(Length(l)));
end;

```

```

LatticeDecompositions:= function(n)

```

```

local d,ind,pp,p1,subgr,ld;
ind:=[];
for d in [1..n] do
  pp:=Partitions(d);
  p1:=List(pp,SortedList);
  if ValueOption("fromperm")=true or ValueOption("FromPerm")=true then
    subgr:=List(p1,x->
      Concatenation(List(AllDirectProductIndmfMatrixGroups(x),
        y->ConjugacyClassesSubgroupsFromPerm(y))));
  else
    subgr:=List(p1,x->
      Concatenation(List(AllDirectProductIndmfMatrixGroups(x),
        y->List(ConjugacyClassesSubgroups2(y),Representative))));
  fi;
  subgr:=List([1..NrPartitions(d)],
    x->Filtered(subgr[x],y->ForAll([1..Length(p1[x])],z->
      p1[x][z]=1 or p1[x][z]=d or
      CaratZClass(PartialMatrixGroup(y,
        [Sum([1..z-1],w->p1[x][w])+1..Sum([1..z],w->p1[x][w]))]
        in ind[p1[x][z]]))););
  ld:=List(subgr,x->Set(x,CaratZClass));
  ld[NrPartitions(d)]:=Difference(ld[NrPartitions(d)],
    Union(List([1..NrPartitions(d)-1],x->ld[x])));
  ind[d]:=ld[NrPartitions(d)];
od;
if ValueOption("carat")=true or ValueOption("Carat")=true then
  return ld;
else
  return List(ld,x->Set(x,Carat2CrystCat));
fi;
end;

```

**Example 4.8** (`DirectSumMatrixGroup(l)`, `DirectProductMatrixGroup(l)`).

`DirectSumMatrixGroup(l)` returns the direct sum of the groups  $G_1, \dots, G_n$  for the list  $l = [G_1, \dots, G_n]$ .

`DirectProductMatrixGroup(l)` returns the direct product of the groups  $G_1, \dots, G_n$  for the list  $l = [G_1, \dots, G_n]$ .

```

gap> Read("caratnumber.gap");
gap> Read("KS.gap");

gap> G:=ImfMatrixGroup(1,1,1); # G=C2 of dimension 1
ImfMatrixGroup(1,1,1)
gap> GeneratorsOfGroup(G);
[ [ [ -1 ] ] ]
gap> Gs3:=DirectSumMatrixGroup([G,G,G]); # Gs3=C2 of dimension 3
Group([ [ [ -1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, -1 ] ] ])
gap> Order(Gs3);
2
gap> Gp3:=DirectProductMatrixGroup([G,G,G]); # Gp3=C2^3 of dimension 3
<matrix group with 3 generators>
gap> GeneratorsOfGroup(Gp3);
[ [ [ -1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
  [ [ 1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, 1 ] ],
  [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, -1 ] ] ]
gap> Size(Gp3);

```

**Example 4.9** (Algorithm KS1). By Algorithm KS1, we may check that the condition (KS1) holds when  $\text{rank } M_G = n \leq 4$  and fails for the 11 cases when  $\text{rank } M_G = 5$  and the 131 cases when  $\text{rank } M_G = 6$  as in Theorem 4.6. The decomposition types of the 11  $G$ -lattices  $M_G$  of rank 5 are  $(3, 2)$  and  $(4, 1)$  and  $G$  is a subgroup of the group  $C_2 \times D_6$  of the CARAT code  $(5, 205, 6)$ . The decomposition types of the former 51 cases (resp. the latter 80 cases) out of the 131 cases are  $(3, 2, 1)$  and  $(4, 1, 1)$  (resp.  $(3, 3)$  and  $(5, 1)$ ) and  $G$  is a subgroup of the group  $C_2^2 \times D_6$  of the CARAT code  $(6, 2139, 9)$  (resp.  $D_6 \times D_4$  of the CARAT code  $(6, 145, 4)$ ). For the groups  $G_1$  and  $G_2$  of the CARAT codes  $(6, 2139, 9)$  and  $(5, 205, 6)$ , we see that  $M_{G_1} \simeq M_{G_2} \oplus N$  for some  $G$ -lattice  $N$  of rank 1.

The computation in this example also confirms that the 2 (resp. 3, 6, 7) irreducible maximal finite subgroups  $\text{Imf}(n, i, j)$  of dimension 2 (resp. 3, 4, 5) become the indecomposable maximal finite subgroups, and the 18 groups given in Subsection 4.0 are the indecomposable maximal finite subgroups of dimension 6 (see also Example 8.2). Indeed, we get the 18996 indecomposable (conjugacy classes of) subgroups of  $\text{GL}(6, \mathbb{Z})$  as the subgroups of the 18 indecomposable maximal finite groups although we obtain only 14348 subgroups if we use the 17 irreducible ones.

```
gap> Read("caratnumber.gap");
gap> Read("KS.gap");

gap> ld1:=LatticeDecompositions(1);
[ [ [ 1, 1, 1 ], [ 1, 2, 1 ] ] ]

gap> ld2:=LatticeDecompositions(2);
[ [ [ 2, 1, 1, 1 ], [ 2, 1, 2, 1 ], [ 2, 2, 1, 1 ], [ 2, 2, 2, 1 ] ],
  [ [ 2, 2, 1, 2 ], [ 2, 2, 2, 2 ], [ 2, 3, 1, 1 ], [ 2, 3, 2, 1 ], [ 2, 4, 1, 1 ],
    [ 2, 4, 2, 1 ], [ 2, 4, 2, 2 ], [ 2, 4, 3, 1 ], [ 2, 4, 4, 1 ] ] ]
gap> Partitions(2);
[ [ 1, 1 ], [ 2 ] ]
gap> List(ld2, Length);
[ 4, 9 ]
gap> [Length(Union(ld2)), Sum(ld2, Length)];
[ 13, 13 ]

gap> ld3:=LatticeDecompositions(3);
[ [ [ 3, 1, 1, 1 ], [ 3, 1, 2, 1 ], [ 3, 2, 1, 1 ], [ 3, 2, 2, 1 ], [ 3, 2, 3, 1 ],
    [ 3, 3, 1, 1 ], [ 3, 3, 2, 1 ], [ 3, 3, 3, 1 ] ],
  [ [ 3, 2, 1, 2 ], [ 3, 2, 2, 2 ], [ 3, 2, 3, 2 ], [ 3, 3, 1, 2 ], [ 3, 3, 2, 2 ],
    [ 3, 3, 2, 3 ], [ 3, 3, 3, 2 ], [ 3, 4, 1, 1 ], [ 3, 4, 2, 1 ], [ 3, 4, 3, 1 ],
    [ 3, 4, 4, 1 ], [ 3, 4, 5, 1 ], [ 3, 4, 6, 1 ], [ 3, 4, 6, 2 ], [ 3, 4, 7, 1 ],
    [ 3, 5, 1, 2 ], [ 3, 5, 2, 2 ], [ 3, 5, 3, 2 ], [ 3, 5, 3, 3 ], [ 3, 5, 4, 2 ],
    [ 3, 5, 4, 3 ], [ 3, 5, 5, 2 ], [ 3, 5, 5, 3 ], [ 3, 6, 1, 1 ], [ 3, 6, 2, 1 ],
    [ 3, 6, 3, 1 ], [ 3, 6, 4, 1 ], [ 3, 6, 5, 1 ], [ 3, 6, 6, 1 ], [ 3, 6, 6, 2 ],
    [ 3, 6, 7, 1 ] ],
  [ [ 3, 3, 1, 3 ], [ 3, 3, 1, 4 ], [ 3, 3, 2, 4 ], [ 3, 3, 2, 5 ], [ 3, 3, 3, 3 ],
    [ 3, 3, 3, 4 ], [ 3, 4, 1, 2 ], [ 3, 4, 2, 2 ], [ 3, 4, 3, 2 ], [ 3, 4, 4, 2 ],
    [ 3, 4, 5, 2 ], [ 3, 4, 6, 3 ], [ 3, 4, 6, 4 ], [ 3, 4, 7, 2 ], [ 3, 5, 1, 1 ],
    [ 3, 5, 2, 1 ], [ 3, 5, 3, 1 ], [ 3, 5, 4, 1 ], [ 3, 5, 5, 1 ], [ 3, 7, 1, 1 ],
    [ 3, 7, 1, 2 ], [ 3, 7, 1, 3 ], [ 3, 7, 2, 1 ], [ 3, 7, 2, 2 ], [ 3, 7, 2, 3 ],
    [ 3, 7, 3, 1 ], [ 3, 7, 3, 2 ], [ 3, 7, 3, 3 ], [ 3, 7, 4, 1 ], [ 3, 7, 4, 2 ],
    [ 3, 7, 4, 3 ], [ 3, 7, 5, 1 ], [ 3, 7, 5, 2 ], [ 3, 7, 5, 3 ] ] ]
gap> Partitions(3);
[ [ 1, 1, 1 ], [ 2, 1 ], [ 3 ] ]
gap> List(ld3, Length);
[ 8, 31, 34 ]
```

```

gap> [Length(Union(ld3)),Sum(ld3,Length)];
[ 73, 73 ]

gap> ld4:=LatticeDecompositions(4);;
gap> Partitions(4);
[ [ 1, 1, 1, 1 ], [ 2, 1, 1 ], [ 2, 2 ], [ 3, 1 ], [ 4 ] ]
gap> List(ld4,Length);
[ 16, 96, 175, 128, 295 ]
gap> [Length(Union(ld4)),Sum(ld4,Length)];
[ 710, 710 ]

gap> ld5:=LatticeDecompositions(5:Carat);;
gap> Partitions(5);
[ [ 1, 1, 1, 1, 1 ], [ 2, 1, 1, 1 ], [ 2, 2, 1 ], [ 3, 1, 1 ], [ 3, 2 ], [ 4, 1 ], [ 5 ] ]
gap> List(ld5,Length);
[ 32, 280, 1004, 442, 1480, 1400, 1452 ]
gap> [Length(Union(ld5)),Sum(ld5,Length)];
[ 6079, 6090 ]
gap> last[2]-last[1];
11
gap> f32_41:=Intersection(ld5[5],ld5[6]); # type [ 3, 2 ] and [ 4, 1 ]
[ [ 5, 188, 4 ], [ 5, 189, 4 ], [ 5, 190, 6 ], [ 5, 191, 6 ], [ 5, 192, 6 ],
  [ 5, 193, 4 ], [ 5, 205, 6 ], [ 5, 218, 8 ], [ 5, 219, 8 ], [ 5, 220, 4 ],
  [ 5, 221, 4 ] ]
gap> Length(f32_41);
11

gap> f32_41g:=List(f32_41,x->CaratMatGroupZClass(x[1],x[2],x[3]));;
gap> List(f32_41g,Order);
[ 12, 12, 12, 12, 12, 12, 24, 6, 6, 6, 6 ]
gap> List(f32_41g,StructureDescription);
[ "D12", "C6 x C2", "D12", "D12", "D12", "D12", "C2 x C2 x S3", "S3", "S3", "C6", "C6" ]
gap> last[7]; # the 7th group of the CARAT code [ 5, 205, 6 ] is maximal
"C2 x C2 x S3"
gap> f32_41gsub:=Set(ConjugacyClassesSubgroups2(f32_41g[7]),
> x->CaratZClass(Representative(x)));;
gap> Difference(f32_41,f32_41gsub);
[ ]
gap> f32_41[7];
[ 5, 205, 6 ]

gap> ld6:=LatticeDecompositions(6:Carat,FromPerm);;
gap> Partitions(6);
[ [ 1, 1, 1, 1, 1, 1 ], [ 2, 1, 1, 1, 1 ], [ 2, 2, 1, 1 ], [ 2, 2, 2 ],
  [ 3, 1, 1, 1 ], [ 3, 2, 1 ], [ 3, 3 ], [ 4, 1, 1 ], [ 4, 2 ], [ 5, 1 ], [ 6 ] ]
gap> List(ld6,Length);
[ 68, 824, 4862, 6878, 1466, 10662, 4235, 5944, 21573, 9931, 18996 ]
gap> [Length(Union(ld6)),Sum(ld6,Length)];
[ 85308, 85439 ]
gap> last[2]-last[1];
131
gap> f321_411:=Intersection(ld6[6],ld6[8]); # type [ 3, 2, 1 ] and [ 4, 1, 1 ]
[ [ 6, 2013, 8 ], [ 6, 2018, 4 ], [ 6, 2023, 6 ], [ 6, 2024, 6 ], [ 6, 2025, 6 ],
  [ 6, 2026, 6 ], [ 6, 2033, 6 ], [ 6, 2042, 8 ], [ 6, 2043, 8 ], [ 6, 2044, 4 ],
  [ 6, 2045, 4 ], [ 6, 2048, 5 ], [ 6, 2049, 8 ], [ 6, 2050, 8 ], [ 6, 2051, 8 ],

```

```

[ 6, 2052, 8 ], [ 6, 2058, 5 ], [ 6, 2059, 5 ], [ 6, 2067, 5 ], [ 6, 2068, 5 ],
[ 6, 2069, 5 ], [ 6, 2069, 11 ], [ 6, 2070, 9 ], [ 6, 2071, 9 ], [ 6, 2072, 10 ],
[ 6, 2072, 11 ], [ 6, 2076, 24 ], [ 6, 2076, 25 ], [ 6, 2077, 24 ], [ 6, 2077, 25 ],
[ 6, 2078, 24 ], [ 6, 2078, 25 ], [ 6, 2079, 24 ], [ 6, 2079, 25 ], [ 6, 2087, 15 ],
[ 6, 2088, 15 ], [ 6, 2089, 17 ], [ 6, 2089, 18 ], [ 6, 2094, 9 ], [ 6, 2102, 24 ],
[ 6, 2102, 25 ], [ 6, 2105, 9 ], [ 6, 2106, 9 ], [ 6, 2107, 10 ], [ 6, 2107, 11 ],
[ 6, 2108, 15 ], [ 6, 2109, 15 ], [ 6, 2110, 17 ], [ 6, 2110, 18 ], [ 6, 2111, 15 ],
[ 6, 2139, 9 ] ]
gap> Length(f321_411);
51

gap> f321_411g:=List(f321_411,x->CaratMatGroupZClass(x[1],x[2],x[3]));;
gap> List(f321_411g,Order);
[ 12, 12, 12, 12, 12, 12, 24, 6, 6, 6, 6, 12, 12, 12, 12, 12, 12, 24, 6, 6, 6, 6, 12, 12,
  12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 24, 24, 24, 24, 24, 24,
  24, 24, 24, 24, 48 ]
gap> List(f321_411g,StructureDescription);
[ "D12", "C6 x C2", "D12", "D12", "D12", "D12", "C2 x C2 x S3", "S3", "S3", "C6", "C6",
  "C6 x C2", "D12", "D12", "D12", "D12", "D12", "C2 x C2 x S3", "C6", "C6", "S3", "S3",
  "C6 x C2", "C6 x C2", "C6 x C2", "C6 x C2", "D12", "D12", "D12", "D12", "D12", "D12",
  "D12", "D12", "D12", "D12", "D12", "D12", "C6 x C2 x C2", "C2 x C2 x S3", "C2 x C2 x S3",
  "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3",
  "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x C2 x S3" ]
gap> last[51]; # the 51th group of the CARAT code [ 6, 2139, 9 ] is maximal
"C2 x C2 x C2 x S3"
gap> f321_411gsub:=Set(ConjugacyClassesSubgroups2(f321_411g[51]),
> x->CaratZClass(Representative(x)));;
gap> Difference(f321_411,f321_411gsub);
[ ]
gap> f321_411[51];
[ 6, 2139, 9 ]

gap> gg:=GeneratorsOfGroup(CaratMatGroupZClass(5,205,6));;
gap> gg:=List(gg,x->DirectSumMat([[1]],x));;
gap> Add(gg,-IdentityMat(6));
gap> CaratZClass(Group(gg));
[ 6, 2139, 9 ]

gap> f33_51:=Intersection(ld6[7],ld6[10]); # type [ 3, 3 ] and [ 5, 1 ]
[ [ 6, 40, 4 ], [ 6, 41, 4 ], [ 6, 44, 6 ], [ 6, 45, 6 ], [ 6, 47, 4 ],
  [ 6, 53, 4 ], [ 6, 54, 4 ], [ 6, 54, 8 ], [ 6, 55, 4 ], [ 6, 63, 4 ],
  [ 6, 64, 6 ], [ 6, 65, 4 ], [ 6, 66, 6 ], [ 6, 67, 6 ], [ 6, 75, 4 ],
  [ 6, 75, 8 ], [ 6, 76, 8 ], [ 6, 76, 12 ], [ 6, 77, 8 ], [ 6, 77, 12 ],
  [ 6, 78, 4 ], [ 6, 78, 8 ], [ 6, 79, 6 ], [ 6, 80, 4 ], [ 6, 81, 8 ],
  [ 6, 81, 12 ], [ 6, 90, 4 ], [ 6, 99, 4 ], [ 6, 108, 4 ], [ 6, 108, 8 ],
  [ 6, 109, 8 ], [ 6, 109, 12 ], [ 6, 110, 4 ], [ 6, 111, 6 ], [ 6, 112, 8 ],
  [ 6, 112, 12 ], [ 6, 113, 4 ], [ 6, 114, 6 ], [ 6, 115, 6 ], [ 6, 145, 4 ],
  [ 6, 2070, 10 ], [ 6, 2070, 11 ], [ 6, 2071, 10 ], [ 6, 2071, 11 ], [ 6, 2072, 12 ],
  [ 6, 2072, 13 ], [ 6, 2076, 26 ], [ 6, 2076, 27 ], [ 6, 2077, 26 ], [ 6, 2077, 27 ],
  [ 6, 2078, 26 ], [ 6, 2078, 27 ], [ 6, 2079, 26 ], [ 6, 2079, 27 ], [ 6, 2087, 16 ],
  [ 6, 2087, 17 ], [ 6, 2088, 16 ], [ 6, 2088, 17 ], [ 6, 2089, 19 ], [ 6, 2089, 20 ],
  [ 6, 2094, 10 ], [ 6, 2094, 11 ], [ 6, 2102, 26 ], [ 6, 2102, 27 ], [ 6, 2105, 10 ],
  [ 6, 2105, 11 ], [ 6, 2106, 10 ], [ 6, 2106, 11 ], [ 6, 2107, 12 ], [ 6, 2107, 13 ],
  [ 6, 2108, 16 ], [ 6, 2108, 17 ], [ 6, 2109, 16 ], [ 6, 2109, 17 ], [ 6, 2110, 19 ],
  [ 6, 2110, 20 ], [ 6, 2111, 16 ], [ 6, 2111, 17 ], [ 6, 2139, 10 ], [ 6, 2139, 11 ] ]

```

```

gap> Length(f33_51);
80

gap> f33_51g:=List(f33_51,x->CaratMatGroupZClass(x[1],x[2],x[3]));;
gap> List(f33_51g,Order);
[ 12, 12, 12, 12, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24,
  24, 24, 24, 24, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 96, 12, 12, 12, 12,
  12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 24, 24, 24, 24, 24, 24,
  24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 48, 48 ]
gap> List(f33_51g,StructureDescription);
[ "C12", "C12", "C3 : C4", "C3 : C4", "C12 x C2", "C3 x D8", "C3 x D8", "C3 x D8",
  "C3 x D8", "C4 x S3", "C4 x S3", "C4 x S3", "C4 x S3", "C2 x (C3 : C4)", "(C6 x C2) : C2",
  "(C6 x C2) : C2", "(C6 x C2) : C2", "(C6 x C2) : C2", "(C6 x C2) : C2", "(C6 x C2) : C2",
  "(C6 x C2) : C2", "D24", "D24", "D24", "D24", "C6 x D8", "C2 x C4 x S3",
  "D8 x S3", "D8 x S3", "C2 x ((C6 x C2) : C2)", "C2 x ((C6 x C2) : C2)", "D8 x S3",
  "D8 x S3", "D8 x S3", "D8 x S3", "D8 x S3", "D8 x S3", "C2 x D24", "C2 x S3 x D8",
  "C6 x C2", "C6 x C2", "C6 x C2", "C6 x C2", "C6 x C2", "C6 x C2", "D12", "D12", "D12",
  "D12", "D12", "D12", "D12", "D12", "D12", "D12", "D12", "D12", "D12",
  "C6 x C2 x C2", "C6 x C2 x C2", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3",
  "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3",
  "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3",
  "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x C2 x S3", "C2 x C2 x C2 x S3" ]
gap> last[40]; # the 40th group of the CARAT code [ 6, 145, 4 ] is maximal
"C2 x S3 x D8"
gap> f33_51gsub:=Set(ConjugacyClassesSubgroups2(f33_51g[40]),
> x->CaratZClass(Representative(x)));;
gap> Difference(f33_51,f33_51gsub);
[ ]
gap> f33_51[40];
[ 6, 145, 4 ]

```

**Example 4.10** (The generators of the exceptional 11 groups  $G \leq \text{GL}(5, \mathbb{Z})$  whose corresponding  $G$ -lattices  $M \simeq M_1 \oplus M_2 \simeq N_1 \oplus N_2$  with  $\text{rank } M_1 = 4$ ,  $\text{rank } M_2 = 1$ ,  $\text{rank } N_1 = 3$  and  $\text{rank } N_2 = 2$ ). Let  $I$  be the identity matrix of rank  $n$ . Let

$$X = \left( \begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right), \quad Y = \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Then the CARAT code and the generators of the exceptional 11 groups  $G \leq \text{GL}(5, \mathbb{Z})$  are given as

$G$	CARAT code	Generators
$S_3 \times C_2 \simeq D_6$	(5, 188, 4)	$\langle X^2, XY, -I \rangle$
$C_2 \times C_6$	(5, 189, 4)	$\langle X, -I \rangle$
$D_6$	(5, 190, 6)	$\langle -X, Y \rangle$
$D_6$	(5, 191, 6)	$\langle -X, XY \rangle$
$D_6$	(5, 192, 6)	$\langle X, Y \rangle$
$D_6$	(5, 193, 4)	$\langle X, -Y \rangle$
$D_6 \times C_2$	(5, 205, 6)	$\langle X, Y, -I \rangle$
$S_3$	(5, 218, 8)	$\langle X^2, XY \rangle$
$S_3$	(5, 219, 8)	$\langle X^2, -XY \rangle$
$C_6$	(5, 220, 4)	$\langle X \rangle$
$C_6$	(5, 221, 4)	$\langle -X \rangle$

The group  $\langle X, Y \rangle \leq GL(5, \mathbb{Z})$  may be regarded as the group embedded directly by the group  $G \leq GL(4, \mathbb{Z})$  of the GAP code (4, 14, 8, 2). We also see that

$$P^{-1}XP = \left( \begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right), \quad P^{-1}YP = \left( \begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad \text{where } P = \left( \begin{array}{ccccc} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

This shows that  $M \simeq M_1 \oplus M_2 \simeq N_1 \oplus N_2$  with  $\text{rank } M_1 = 4$ ,  $\text{rank } M_2 = 1$ ,  $\text{rank } N_1 = 3$  and  $\text{rank } N_2 = 2$ .

**4.2. Krull-Schmidt theorem (2).** We will determine whether the condition (KS2) holds.

**Algorithm KS2** (Determination whether the condition (KS2) holds).

(The following algorithm needs the CARAT package of GAP and caratnumber.gap).

```
InverseProjection:= function(l)
  local lc, lcg, lg, ll, G, N, i, j, k, gn, gn1, gn2, o, gN, h, h1, ans, ans1;
  lc:=Collected(l);
  if ValueOption("carat")=true or ValueOption("Carat")=true then
    lcg:=List(lc, x->
      [CaratMatGroupZClass(x[1][1], x[1][2], x[1][3]), x[2]]);
  else
    lcg:=List(lc, x->
      [MatGroupZClass(x[1][1], x[1][2], x[1][3], x[1][4]), x[2]]);
  fi;
  lg:=Concatenation(List(lcg, x->List([1..x[2]], y->x[1])));
  ll:=Concatenation(List(lc, x->List([1..x[2]], y->x[1][1])));
  G:=DirectProductMatrixGroup(lg);
  gn:=[];
  for i in [1..Length(lc)] do
    if lc[i][1][1]=1 then
      gn1:=[[[-1]]];
    else
      gn1:=GeneratorsOfGroup(Normalizer(GL(lc[i][1][1], Integers),
        lcg[i][1]));
    fi;
    gn2:=[];
    for j in [1..lc[i][2]] do
      o:=List([1..lc[i][2]], x->IdentityMat(lc[i][1][1]));
      for k in gn1 do
        o[j]:=k;
        Add(gn2, DirectSumMat(o));
      od;
    od;
    gn1:=GeneratorsOfGroup(SymmetricGroup(lc[i][2]));
    gn2:=Concatenation(gn2, List(gn1,
      x->KroneckerProduct(PermutationMat(x, lc[i][2]),
        IdentityMat(lc[i][1][1]))));
    Add(gn, Group(gn2));
  od;
  N:=DirectProductMatrixGroup(gn);
  ans1:=List(ConjugacyClassesSubgroups2(G), Representative);
  ans1:=Filtered(ans1, x->
    ForAll([1..Length(lg)], y->PartialMatrixGroup(x,
      [Sum([1..y-1], z->ll[z])+1..Sum([1..y], z->ll[z])]=lg[y]));
  ans:=[];
```

```

gN:=GeneratorsOfGroup(N);
while ans1<>[] do
  h:=[];
  h1:=ans1[1];
  while h1<>[] do
    h:=Union(h,h1);
    h1:=Difference(Concatenation(List(h1,x->List(gN,y->x^y))),h);
  od;
  Add(ans,ans1[1]);
  ans1:=Difference(ans1,h);
od;
return ans;
end;

```

**Example 4.11** (`InverseProjection([11,12])`). Let  $G_1 \simeq C_4 \leq \text{GL}(2, \mathbb{Z})$  (resp.  $G_2 \simeq D_6 \leq \text{GL}(2, \mathbb{Z})$ ) be a cyclic group of order 4 (resp. dihedral group of order 12) of the GAP code  $l_1 = (2, 3, 1, 1)$  (resp.  $l_2 = (2, 4, 4, 1)$ ).

`InverseProjection([11,12])` returns the list of all groups  $G \leq \text{GL}(4, \mathbb{Z})$  such that  $M_G \simeq M_{G_1} \oplus M_{G_2}$  and the GAP code of  $G_1$  (resp.  $G_2$ ) is  $l_1$  (resp.  $l_2$ ).

`InverseProjection([11,12]:Carat)` returns the same as `InverseProjection([11,12])` but with respect to the CARAT code  $l_1$  and  $l_2$  instead of the GAP code.

```

Read("crystcat.gap");
Read("caratnumber.gap");
Read("KS.gap");

```

```

gap> pgs:=InverseProjection([[2,3,1,1],[2,4,4,1]]);
[ <matrix group of size 24 with 4 generators>,
  <matrix group of size 48 with 5 generators>,
  <matrix group of size 24 with 4 generators>,
  <matrix group of size 24 with 4 generators> ]
gap> List(pgs,StructureDescription);
[ "C2 x (C3 : C4)", "C2 x C4 x S3", "C4 x S3", "C4 x S3" ]
gap> List(pgs,CrystCatZClass);
[ [ 4, 20, 14, 1 ], [ 4, 20, 15, 1 ], [ 4, 20, 9, 2 ], [ 4, 20, 9, 1 ] ]

```

**Example 4.12** (Verification of  $[M_G]^{fl} \neq 0$  and that  $[M_G]^{fl}$  is not invertible where  $M_G \simeq M_{G_1} \oplus M_{G_2}$  is a decomposable  $G$ -lattice). By using `InverseProjection` in Algorithm KS2, we may obtain Table 3 of Theorems 1.8 and 1.25 and Tables 11 to 14 in Theorems 1.11 and 1.26.

Let  $G \leq \text{GL}(4, \mathbb{Z})$  where  $M_G \simeq M_{G_1} \oplus M_{G_2}$  is a decomposable  $G$ -lattice of rank 4 with  $G_1 \leq \text{GL}(3, \mathbb{Z})$  and  $G_2 \leq \text{GL}(1, \mathbb{Z})$ . Let  $\mathcal{N}_3$  be the set of the 15 GAP codes as in Table 1. By Lemma 2.14, we have  $[M_G]^{fl} = [M_1]^{fl}$ . Hence, by Theorem 1.2 and Theorem 2.12,  $L(M_G)^G$  is not retract  $k$ -rational if and only if  $L(M_G)^G$  is not stably  $k$ -rational if and only if the GAP code of  $G_1$  is  $l \in \mathcal{N}_3$ .

All the GAP codes  $\mathcal{N}_{31}$ , as in Table 3 of Theorem 1.8, of such groups  $G$  which satisfy  $M_G \simeq M_{G_1} \oplus M_{G_2}$ ,  $G_1 \leq \text{GL}(3, \mathbb{Z})$  of the GAP code  $l \in \mathcal{N}_3$  and  $G_2 \leq \text{GL}(1, \mathbb{Z})$  may be obtained as follows.

Similarly, by using the result  $\mathcal{I}_4$  and  $\mathcal{N}_4$  as in Tables 2 and 4 of Theorems 1.8 and 1.25, we may obtain the CARAT codes  $\mathcal{I}_{41}$ ,  $\mathcal{N}_{311}$ ,  $\mathcal{N}_{32}$  and  $\mathcal{N}_{41}$  as in Tables 11 to 14 of Theorems 1.11 and 1.26.

```

Read("crystcat.gap");
Read("caratnumber.gap");
Read("KS.gap");

```

```

gap> ind1:=LatticeDecompositions(1:Carat)[NrPartitions(1)];
[ [ 1, 1, 1 ], [ 1, 2, 1 ] ]

```



```

gap> ind2:=LatticeDecompositions(2:Carat)[NrPartitions(2)];
[ [ 2, 3, 2 ], [ 2, 4, 2 ], [ 2, 5, 1 ], [ 2, 6, 1 ], [ 2, 7, 1 ],
  [ 2, 8, 1 ], [ 2, 9, 1 ], [ 2, 10, 1 ], [ 2, 10, 2 ] ]
gap> ind3:=LatticeDecompositions(3:Carat)[NrPartitions(3)];
[ [ 3, 6, 3 ], [ 3, 6, 4 ], [ 3, 7, 4 ], [ 3, 7, 5 ], [ 3, 8, 3 ], [ 3, 8, 4 ],
  [ 3, 9, 2 ], [ 3, 10, 2 ], [ 3, 11, 2 ], [ 3, 12, 2 ], [ 3, 13, 2 ], [ 3, 14, 2 ],
  [ 3, 14, 4 ], [ 3, 15, 2 ], [ 3, 17, 2 ], [ 3, 22, 2 ], [ 3, 25, 2 ], [ 3, 26, 2 ],
  [ 3, 27, 2 ], [ 3, 28, 1 ], [ 3, 28, 2 ], [ 3, 28, 3 ], [ 3, 29, 1 ], [ 3, 29, 2 ],
  [ 3, 29, 3 ], [ 3, 30, 1 ], [ 3, 30, 2 ], [ 3, 30, 3 ], [ 3, 31, 1 ], [ 3, 31, 2 ],
  [ 3, 31, 3 ], [ 3, 32, 1 ], [ 3, 32, 2 ], [ 3, 32, 3 ] ]
gap> ind4:=LatticeDecompositions(4:Carat)[NrPartitions(4)];
gap> ind5:=LatticeDecompositions(5:Carat)[NrPartitions(5)];
gap> List([ind1,ind2,ind3,ind4,ind5],Length);
[ 2, 9, 34, 295, 1452 ]

gap> N3:=[ [ 3, 3, 1, 3 ], [ 3, 3, 3, 3 ], [ 3, 3, 3, 4 ], [ 3, 4, 3, 2 ], [ 3, 4, 4, 2 ],
> [ 3, 4, 6, 3 ], [ 3, 4, 7, 2 ], [ 3, 7, 1, 2 ], [ 3, 7, 2, 2 ], [ 3, 7, 2, 3 ],
> [ 3, 7, 3, 2 ], [ 3, 7, 3, 3 ], [ 3, 7, 4, 2 ], [ 3, 7, 5, 2 ], [ 3, 7, 5, 3 ] ];;
gap> Length(N3);
15
gap> N3c:=List(N3,CrystCat2Carat);
[ [ 3, 6, 3 ], [ 3, 8, 3 ], [ 3, 8, 4 ], [ 3, 15, 2 ], [ 3, 13, 2 ],
  [ 3, 14, 4 ], [ 3, 9, 2 ], [ 3, 28, 1 ], [ 3, 29, 1 ], [ 3, 29, 3 ],
  [ 3, 31, 1 ], [ 3, 31, 3 ], [ 3, 30, 1 ], [ 3, 32, 1 ], [ 3, 32, 3 ] ]
gap> N31g:=List(Cartesian([N3c,ind1]),x->InverseProjection(x:Carat));
gap> N31:=Set(Concatenation(N31g),CrystCatZClass);
[ [ 4, 4, 3, 6 ], [ 4, 4, 4, 4 ], [ 4, 4, 4, 6 ], [ 4, 5, 1, 9 ], [ 4, 5, 2, 4 ],
  [ 4, 5, 2, 7 ], [ 4, 6, 1, 4 ], [ 4, 6, 1, 8 ], [ 4, 6, 2, 4 ], [ 4, 6, 2, 8 ],
  [ 4, 6, 2, 9 ], [ 4, 6, 3, 3 ], [ 4, 6, 3, 6 ], [ 4, 7, 3, 2 ], [ 4, 7, 4, 3 ],
  [ 4, 7, 5, 2 ], [ 4, 7, 7, 2 ], [ 4, 12, 2, 4 ], [ 4, 12, 3, 7 ], [ 4, 12, 4, 6 ],
  [ 4, 12, 4, 8 ], [ 4, 12, 4, 9 ], [ 4, 12, 5, 6 ], [ 4, 12, 5, 7 ], [ 4, 13, 1, 3 ],
  [ 4, 13, 2, 4 ], [ 4, 13, 3, 4 ], [ 4, 13, 4, 3 ], [ 4, 13, 5, 3 ], [ 4, 13, 6, 3 ],
  [ 4, 13, 7, 6 ], [ 4, 13, 7, 7 ], [ 4, 13, 7, 8 ], [ 4, 13, 8, 3 ], [ 4, 13, 8, 4 ],
  [ 4, 13, 9, 3 ], [ 4, 13, 10, 3 ], [ 4, 24, 1, 5 ], [ 4, 24, 2, 3 ], [ 4, 24, 2, 5 ],
  [ 4, 24, 3, 5 ], [ 4, 24, 4, 3 ], [ 4, 24, 4, 5 ], [ 4, 24, 5, 3 ], [ 4, 24, 5, 5 ],
  [ 4, 25, 1, 2 ], [ 4, 25, 1, 4 ], [ 4, 25, 2, 4 ], [ 4, 25, 3, 2 ], [ 4, 25, 3, 4 ],
  [ 4, 25, 4, 4 ], [ 4, 25, 5, 2 ], [ 4, 25, 5, 4 ], [ 4, 25, 6, 2 ], [ 4, 25, 6, 4 ],
  [ 4, 25, 7, 2 ], [ 4, 25, 7, 4 ], [ 4, 25, 8, 2 ], [ 4, 25, 8, 4 ], [ 4, 25, 9, 4 ],
  [ 4, 25, 10, 2 ], [ 4, 25, 10, 4 ], [ 4, 25, 11, 2 ], [ 4, 25, 11, 4 ] ]
gap> Length(N31);
64

gap> I4:=[ [ 4, 31, 1, 3 ], [ 4, 31, 1, 4 ], [ 4, 31, 2, 2 ], [ 4, 31, 4, 2 ],
> [ 4, 31, 5, 2 ], [ 4, 31, 7, 2 ], [ 4, 33, 2, 1 ] ];;
gap> Length(I4);
7
gap> N4:=[ [ 4, 5, 1, 12 ], [ 4, 5, 2, 5 ], [ 4, 5, 2, 8 ], [ 4, 5, 2, 9 ], [ 4, 6, 1, 6 ],
> [ 4, 6, 1, 11 ], [ 4, 6, 2, 6 ], [ 4, 6, 2, 10 ], [ 4, 6, 2, 12 ], [ 4, 6, 3, 4 ],
> [ 4, 6, 3, 7 ], [ 4, 6, 3, 8 ], [ 4, 12, 2, 5 ], [ 4, 12, 2, 6 ], [ 4, 12, 3, 11 ],
> [ 4, 12, 4, 10 ], [ 4, 12, 4, 11 ], [ 4, 12, 4, 12 ], [ 4, 12, 5, 8 ], [ 4, 12, 5, 9 ],
> [ 4, 12, 5, 10 ], [ 4, 12, 5, 11 ], [ 4, 13, 1, 5 ], [ 4, 13, 2, 5 ], [ 4, 13, 3, 5 ],
> [ 4, 13, 4, 5 ], [ 4, 13, 5, 4 ], [ 4, 13, 5, 5 ], [ 4, 13, 6, 5 ], [ 4, 13, 7, 9 ],
> [ 4, 13, 7, 10 ], [ 4, 13, 7, 11 ], [ 4, 13, 8, 5 ], [ 4, 13, 8, 6 ], [ 4, 13, 9, 4 ],
> [ 4, 13, 9, 5 ], [ 4, 13, 10, 4 ], [ 4, 13, 10, 5 ], [ 4, 18, 1, 3 ], [ 4, 18, 2, 4 ],
> [ 4, 18, 2, 5 ], [ 4, 18, 3, 5 ], [ 4, 18, 3, 6 ], [ 4, 18, 3, 7 ], [ 4, 18, 4, 4 ],

```

```

> [ 4, 18, 4, 5 ], [ 4, 18, 5, 5 ], [ 4, 18, 5, 6 ], [ 4, 18, 5, 7 ], [ 4, 19, 1, 2 ],
> [ 4, 19, 2, 2 ], [ 4, 19, 3, 2 ], [ 4, 19, 4, 3 ], [ 4, 19, 4, 4 ], [ 4, 19, 5, 2 ],
> [ 4, 19, 6, 2 ], [ 4, 22, 1, 1 ], [ 4, 22, 2, 1 ], [ 4, 22, 3, 1 ], [ 4, 22, 4, 1 ],
> [ 4, 22, 5, 1 ], [ 4, 22, 5, 2 ], [ 4, 22, 6, 1 ], [ 4, 22, 7, 1 ], [ 4, 22, 8, 1 ],
> [ 4, 22, 9, 1 ], [ 4, 22, 10, 1 ], [ 4, 22, 11, 1 ], [ 4, 24, 2, 4 ], [ 4, 24, 2, 6 ],
> [ 4, 24, 4, 4 ], [ 4, 24, 5, 4 ], [ 4, 24, 5, 6 ], [ 4, 25, 1, 3 ], [ 4, 25, 2, 3 ],
> [ 4, 25, 2, 5 ], [ 4, 25, 3, 3 ], [ 4, 25, 4, 3 ], [ 4, 25, 5, 3 ], [ 4, 25, 5, 5 ],
> [ 4, 25, 6, 3 ], [ 4, 25, 6, 5 ], [ 4, 25, 7, 3 ], [ 4, 25, 8, 3 ], [ 4, 25, 9, 3 ],
> [ 4, 25, 9, 5 ], [ 4, 25, 10, 3 ], [ 4, 25, 10, 5 ], [ 4, 25, 11, 3 ], [ 4, 25, 11, 5 ],
> [ 4, 29, 1, 1 ], [ 4, 29, 1, 2 ], [ 4, 29, 2, 1 ], [ 4, 29, 3, 1 ], [ 4, 29, 3, 2 ],
> [ 4, 29, 3, 3 ], [ 4, 29, 4, 1 ], [ 4, 29, 4, 2 ], [ 4, 29, 5, 1 ], [ 4, 29, 6, 1 ],
> [ 4, 29, 7, 1 ], [ 4, 29, 7, 2 ], [ 4, 29, 8, 1 ], [ 4, 29, 8, 2 ], [ 4, 29, 9, 1 ],
> [ 4, 32, 1, 2 ], [ 4, 32, 2, 2 ], [ 4, 32, 3, 2 ], [ 4, 32, 4, 2 ], [ 4, 32, 5, 2 ],
> [ 4, 32, 5, 3 ], [ 4, 32, 6, 2 ], [ 4, 32, 7, 2 ], [ 4, 32, 8, 2 ], [ 4, 32, 9, 4 ],
> [ 4, 32, 9, 5 ], [ 4, 32, 10, 2 ], [ 4, 32, 11, 2 ], [ 4, 32, 11, 3 ], [ 4, 32, 12, 2 ],
> [ 4, 32, 13, 3 ], [ 4, 32, 13, 4 ], [ 4, 32, 14, 3 ], [ 4, 32, 14, 4 ], [ 4, 32, 15, 2 ],
> [ 4, 32, 16, 2 ], [ 4, 32, 16, 3 ], [ 4, 32, 17, 2 ], [ 4, 32, 18, 2 ], [ 4, 32, 18, 3 ],
> [ 4, 32, 19, 2 ], [ 4, 32, 19, 3 ], [ 4, 32, 20, 2 ], [ 4, 32, 20, 3 ], [ 4, 32, 21, 2 ],
> [ 4, 32, 21, 3 ], [ 4, 33, 1, 1 ], [ 4, 33, 3, 1 ], [ 4, 33, 4, 1 ], [ 4, 33, 5, 1 ],
> [ 4, 33, 6, 1 ], [ 4, 33, 7, 1 ], [ 4, 33, 8, 1 ], [ 4, 33, 9, 1 ], [ 4, 33, 10, 1 ],
> [ 4, 33, 11, 1 ], [ 4, 33, 12, 1 ], [ 4, 33, 13, 1 ], [ 4, 33, 14, 1 ], [ 4, 33, 14, 2 ],
> [ 4, 33, 15, 1 ], [ 4, 33, 16, 1 ] ];
gap> Length(N4);
152

```

```

gap> I4c:=List(I4,x->CaratZClass(MatGroupZClass(x[1],x[2],x[3],x[4])));
gap> I41g:=List(Cartesian([I4c,ind1]),x->InverseProjection(x:Carat));
gap> I41:=Set(Concatenation(I41g),CaratZClass);
[ [ 5, 692, 1 ], [ 5, 693, 1 ], [ 5, 736, 1 ], [ 5, 911, 1 ], [ 5, 912, 1 ],
  [ 5, 914, 1 ], [ 5, 916, 1 ], [ 5, 917, 1 ], [ 5, 917, 5 ], [ 5, 918, 1 ],
  [ 5, 918, 5 ], [ 5, 919, 1 ], [ 5, 921, 1 ], [ 5, 922, 1 ], [ 5, 923, 1 ],
  [ 5, 924, 1 ], [ 5, 925, 1 ], [ 5, 926, 1 ], [ 5, 926, 3 ], [ 5, 927, 1 ],
  [ 5, 928, 1 ], [ 5, 928, 3 ], [ 5, 929, 1 ], [ 5, 930, 1 ], [ 5, 932, 1 ] ]
gap> Length(I41);
25
gap> N311g:=List(Cartesian([UnorderedTuples(N3c,1),UnorderedTuples(ind1,2)]),
> x->InverseProjection(Concatenation(x):Carat));
gap> N311:=Set(Concatenation(N311g),CaratZClass);
gap> Length(N311);
245
gap> N32g:=List(Cartesian([N3c,ind2]),x->InverseProjection(x:Carat));
gap> N32:=Set(Concatenation(N32g),CaratZClass);
gap> Length(N32);
849
gap> N4c:=List(N4,x->CaratZClass(MatGroupZClass(x[1],x[2],x[3],x[4])));
gap> N41g:=List(Cartesian([N4c,ind1]),x->InverseProjection(x:Carat));
gap> N41:=Set(Concatenation(N41g),CaratZClass);
Length(N41);
768

```

**Example 4.13** (Verification of the condition (KS2)). By using Algorithm KS2, we may check that the condition (KS2) holds for  $n \leq 5$ . For  $n = 6$ , there exist exactly 10 (resp. 8)  $G$ -lattices of decomposition type  $(4, 2)$  (resp.  $(5, 1)$ ) which are decomposable in two different ways  $M_G \simeq M_1 \oplus M_2 \simeq N_1 \oplus N_2$  as follows:

$M_G \simeq M_1 \oplus M_2 \simeq N_1 \oplus N_2$	$M_1$	$M_2$	$N_1$	$N_2$
(6, 2072, 14) $C_2 \times C_6$	(2, 2, 1, 2) $C_2$	(4, 14, 4, 2) $C_2 \times C_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 1, 2) $C_6$
(6, 2076, 28) $D_6$	(2, 2, 1, 2) $C_2$	(4, 14, 9, 2) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 3, 4) $S_3$
(6, 2077, 28) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 5, 2) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 7, 2) $D_6$
(6, 2078, 28) $D_6$	(2, 2, 1, 2) $C_2$	(4, 14, 5, 2) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 6, 2) $D_6$
(6, 2079, 28) $D_6$	(2, 2, 1, 2) $C_2$	(4, 14, 8, 2) $D_6$	(2, 2, 1, 2) $C_2$	(4, 14, 3, 3) $S_3$
(6, 2089, 21) $D_6$	(2, 2, 1, 2) $C_2$	(4, 14, 7, 2) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 6, 2) $D_6$
(6, 2102, 28) $C_2 \times D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 5, 2) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 10, 2) $C_2 \times D_6$
(6, 2107, 14) $C_2 \times D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 6, 2) $D_6$	(2, 2, 1, 2) $C_2$	(4, 14, 10, 2) $C_2 \times D_6$
(6, 2110, 21) $C_2 \times D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 7, 2) $D_6$	(2, 2, 2, 2) $C_2^2$	(4, 14, 10, 2) $C_2 \times D_6$
(6, 2295, 2) $D_6$	(2, 4, 2, 2) $S_3$	(4, 21, 3, 1) $D_6$	(2, 4, 2, 1) $S_3$	(4, 21, 3, 2) $D_6$

$M_G \simeq M_1 \oplus M_2 \simeq N_1 \oplus N_2$	$M_1$	$M_2$	$N_1$	$N_2$
(6, 3045, 3) $C_2 \times A_5$	(1, 2, 1) $C_2$	(5, 910, 3) $C_2 \times A_5$	(1, 2, 1) $C_2$	(5, 910, 4) $C_2 \times A_5$
(6, 3046, 3) $S_5$	(1, 1, 1) $\{1\}$	(5, 911, 3) $S_5$	(1, 1, 1) $\{1\}$	(5, 911, 4) $S_5$
(6, 3047, 3) $S_5$	(1, 2, 1) $C_2$	(5, 912, 3) $S_5$	(1, 2, 1) $C_2$	(5, 912, 4) $S_5$
(6, 3052, 5) $F_{20}$	(1, 2, 1) $C_2$	(5, 917, 3) $F_{20}$	(1, 2, 1) $C_2$	(5, 917, 4) $F_{20}$
(6, 3053, 5) $F_{20}$	(1, 1, 1) $\{1\}$	(5, 918, 3) $F_{20}$	(1, 1, 1) $\{1\}$	(5, 918, 4) $F_{20}$
(6, 3054, 3) $C_2 \times S_5$	(1, 2, 1) $C_2$	(5, 919, 3) $C_2 \times S_5$	(1, 2, 1) $C_2$	(5, 919, 4) $C_2 \times S_5$
(6, 3061, 5) $C_2 \times F_{20}$	(1, 2, 1) $C_2$	(5, 926, 5) $C_2 \times F_{20}$	(1, 2, 1) $C_2$	(5, 926, 6) $C_2 \times F_{20}$
(6, 3066, 3) $A_5$	(1, 1, 1) $\{1\}$	(5, 931, 3) $A_5$	(1, 1, 1) $\{1\}$	(5, 931, 4) $A_5$

In particular, we see that the 3  $G$ -lattices  $M_G$  for the groups  $S_5$ ,  $F_{20}$  and  $A_5$  of the CARAT codes (5, 911, 4), (5, 918, 4) and (5, 931, 4) respectively are not permutation but stably permutation because the  $G$ -lattices  $M_G$  for the groups  $S_5$ ,  $F_{20}$  and  $A_5$  of the CARAT codes (5, 911, 3), (5, 918, 3) and (5, 931, 3) are permutation. We will study this 3 cases in Section 6 again (see Theorem 6.2 (iii), Table 8 and Example 6.5).

```

Read("crystcat.gap");
Read("caratnumber.gap");
Read("KS.gap");

gap> ind1:=LatticeDecompositions(1:Carat)[NrPartitions(1)];;
gap> ind2:=LatticeDecompositions(2:Carat)[NrPartitions(2)];;
gap> ind3:=LatticeDecompositions(3:Carat)[NrPartitions(3)];;
gap> ind4:=LatticeDecompositions(4:Carat)[NrPartitions(4)];;
gap> ind5:=LatticeDecompositions(5:Carat)[NrPartitions(5)];;
gap> List([ind1,ind2,ind3,ind4,ind5],Length);
[ 2, 9, 34, 295, 1452 ]

gap> ips21:=List(Cartesian([ind2,ind1]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 31, 31 ]

gap> ips211:=List(Cartesian([UnorderedTuples(ind2,1),UnorderedTuples(ind1,2)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 96, 96 ]

gap> ips22:=List(UnorderedTuples(ind2,2),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 175, 175 ]

gap> ips31:=List(Cartesian([ind3,ind1]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 128, 128 ]

gap> ips2111:=List(Cartesian([UnorderedTuples(ind2,1),UnorderedTuples(ind1,3)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);

```

```

[ 280, 280 ]
gap> ips221:=List(Cartesian([UnorderedTuples(ind2,2),UnorderedTuples(ind1,1)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 1004, 1004 ]
gap> ips311:=List(Cartesian([UnorderedTuples(ind3,1),UnorderedTuples(ind1,2)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 442, 442 ]
gap> ips32:=List(Cartesian([ind3,ind2]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 1480, 1480 ]
gap> ips41:=List(Cartesian([ind4,ind1]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 1400, 1400 ]

gap> ips21111:= List(Cartesian([UnorderedTuples(ind2,1),UnorderedTuples(ind1,4)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 824, 824 ]
gap> ips2211:= List(Cartesian([UnorderedTuples(ind2,2),UnorderedTuples(ind1,2)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 4862, 4862 ]
gap> ips222:= List(UnorderedTuples(ind2,3),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 6878, 6878 ]
gap> ips3111:= List(Cartesian([UnorderedTuples(ind3,1),UnorderedTuples(ind1,3)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 1466, 1466 ]
gap> ips321:= List(Cartesian([ind3,ind2,ind1]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 10662, 10662 ]
gap> ips33:= List(UnorderedTuples(ind3,2),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 4235, 4235 ]
gap> ips411:= List(Cartesian([UnorderedTuples(ind4,1),UnorderedTuples(ind1,2)]),
> x->InverseProjection(Concatenation(x):Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 5944, 5944 ]
gap> ips42:= List(Cartesian([ind4,ind2]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 21583, 21573 ]
gap> last[1]-last[2];
10
gap> ips51:= List(Cartesian([ind5,ind1]),x->InverseProjection(x:Carat));;
gap> List([Flat(last),Set(List(Flat(last),CaratZClass))],Length);
[ 9939, 9931 ]
gap> last[1]-last[2];
8

gap> Filtered(Collected(List(Flat(ips42),CaratZClass)),x->x[2]>1); # type [4,2]
[ [ [ 6, 2072, 14 ], 2 ], [ [ 6, 2076, 28 ], 2 ], [ [ 6, 2077, 28 ], 2 ],
  [ [ 6, 2078, 28 ], 2 ], [ [ 6, 2079, 28 ], 2 ], [ [ 6, 2089, 21 ], 2 ],

```

```

[ [ 6, 2102, 28 ], 2 ], [ [ 6, 2107, 14 ], 2 ], [ [ 6, 2110, 21 ], 2 ],
[ [ 6, 2295, 2 ], 2 ] ]
gap> KSfail42:= List(last,x->x[1]);
[ [ 6, 2072, 14 ], [ 6, 2076, 28 ], [ 6, 2077, 28 ], [ 6, 2078, 28 ], [ 6, 2079, 28 ],
[ 6, 2089, 21 ], [ 6, 2102, 28 ], [ 6, 2107, 14 ], [ 6, 2110, 21 ], [ 6, 2295, 2 ] ]
gap> Length(KSfail42);
10
gap> KSfail42g:=List(KSfail42,x->CaratMatGroupZClass(x[1],x[2],x[3]));;
gap> List(KSfail42g,Order);
[ 12, 12, 12, 12, 12, 12, 24, 24, 24, 12 ]
gap> List(KSfail42g,StructureDescription);
[ "C6 x C2", "D12", "D12", "D12", "D12", "D12", "C2 x C2 x S3", "C2 x C2 x S3",
"C2 x C2 x S3", "D12" ]
gap> [last[7],last[8],last[9],last[10]]; # 7th, 8th, 9th, 10th groups are maximal
[ "C2 x C2 x S3", "C2 x C2 x S3", "C2 x C2 x S3", "D12" ]
gap> KSfail42gsub1:=Set(ConjugacyClassesSubgroups2(KSfail42g[7]),
> x->CaratZClass(Representative(x)));;
gap> KSfail42gsub2:=Set(ConjugacyClassesSubgroups2(KSfail42g[8]),
> x->CaratZClass(Representative(x)));;
gap> KSfail42gsub3:=Set(ConjugacyClassesSubgroups2(KSfail42g[9]),
> x->CaratZClass(Representative(x)));;
gap> List(KSfail42,x->x in KSfail42gsub1);
[ false, true, true, true, true, false, true, false, false, false ]
gap> List(KSfail42,x->x in KSfail42gsub2);
[ true, true, false, true, false, true, false, true, false, false ]
gap> List(KSfail42,x->x in KSfail42gsub3);
[ true, false, true, false, true, true, false, false, true, false ]
gap> ips42f:=Flat(ips42);;
gap> ips42fc:=List(ips42f,CaratZClass);;
gap> KSfail42i:=List(KSfail42,x->Filtered([1..21583],y->ips42fc[y]=x));
[ [ 9822, 10444 ], [ 10162, 10496 ], [ 9901, 9960 ], [ 9890, 10028 ], [ 10095, 10518 ],
[ 9956, 10033 ], [ 9895, 10257 ], [ 10030, 10230 ], [ 9963, 10248 ], [ 10647, 10720 ] ]
gap> KSfail42gg:=List(KSfail42i,x->[ips42f[x[1]],ips42f[x[2]]]);;
gap> List(KSfail42gg,x->List(x,y->List([1,2],[3..6]),
> z->CrystCatZClass(PartialMatrixGroup(y,z))));
[ [ [ [ 2, 2, 1, 2 ], [ 4, 14, 4, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 1, 2 ] ] ],
[ [ [ 2, 2, 1, 2 ], [ 4, 14, 9, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 3, 4 ] ] ],
[ [ [ 2, 2, 2, 2 ], [ 4, 14, 5, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 7, 2 ] ] ],
[ [ [ 2, 2, 1, 2 ], [ 4, 14, 5, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 6, 2 ] ] ],
[ [ [ 2, 2, 1, 2 ], [ 4, 14, 8, 2 ] ], [ [ 2, 2, 1, 2 ], [ 4, 14, 3, 3 ] ] ],
[ [ [ 2, 2, 1, 2 ], [ 4, 14, 7, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 6, 2 ] ] ],
[ [ [ 2, 2, 2, 2 ], [ 4, 14, 5, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 10, 2 ] ] ],
[ [ [ 2, 2, 2, 2 ], [ 4, 14, 6, 2 ] ], [ [ 2, 2, 1, 2 ], [ 4, 14, 10, 2 ] ] ],
[ [ [ 2, 2, 2, 2 ], [ 4, 14, 7, 2 ] ], [ [ 2, 2, 2, 2 ], [ 4, 14, 10, 2 ] ] ],
[ [ [ 2, 4, 2, 2 ], [ 4, 21, 3, 1 ] ], [ [ 2, 4, 2, 1 ], [ 4, 21, 3, 2 ] ] ] ]
gap> List(last,y->List([y[1][1],y[1][2],y[2][1],y[2][2]],
> x->StructureDescription(MatGroupZClass(x[1],x[2],x[3],x[4]))));
[ [ "C2", "C6 x C2", "C2 x C2", "C6" ],
[ "C2", "D12", "C2 x C2", "S3" ],
[ "C2 x C2", "D12", "C2 x C2", "D12" ],
[ "C2", "D12", "C2 x C2", "D12" ],
[ "C2", "D12", "C2", "S3" ],
[ "C2", "D12", "C2 x C2", "D12" ],
[ "C2 x C2", "D12", "C2 x C2", "C2 x C2 x S3" ],
[ "C2 x C2", "D12", "C2", "C2 x C2 x S3" ],

```

```

[ "C2 x C2", "D12", "C2 x C2", "C2 x C2 x S3" ],
[ "S3", "D12", "S3", "D12" ] ]

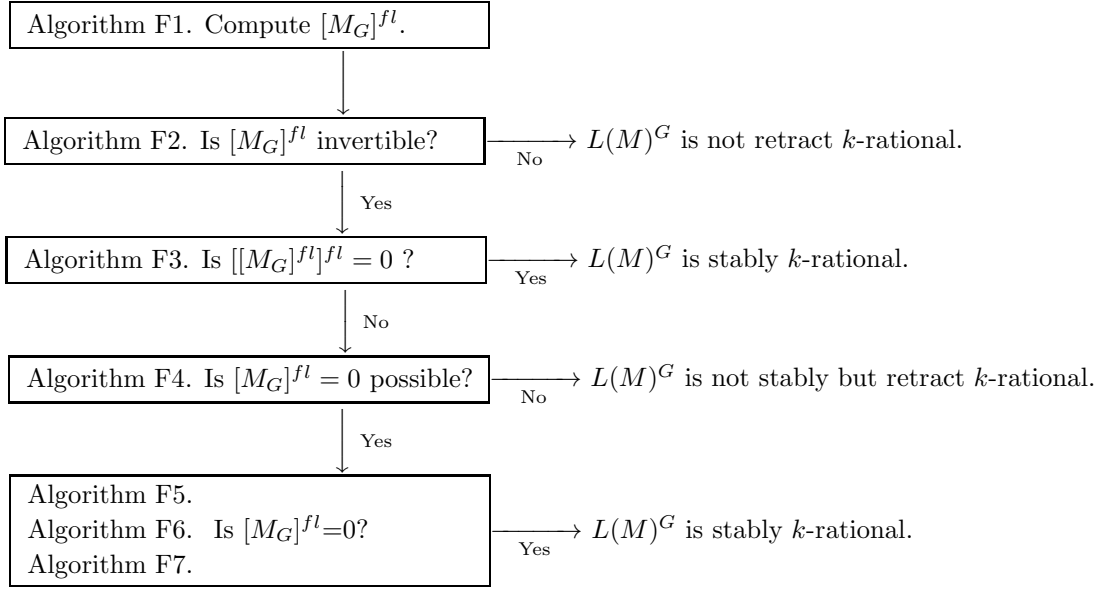
gap> Filtered(Collected(List(Flat(ips51),CaratZClass)),x->x[2]>1); # type [5,1]
[ [ [ 6, 3045, 3 ], 2 ], [ [ 6, 3046, 3 ], 2 ], [ [ 6, 3047, 3 ], 2 ],
  [ [ 6, 3052, 5 ], 2 ], [ [ 6, 3053, 5 ], 2 ], [ [ 6, 3054, 3 ], 2 ],
  [ [ 6, 3061, 5 ], 2 ], [ [ 6, 3066, 3 ], 2 ] ]
gap> KSfail51:= List(last,x->x[1]);
[ [ 6, 3045, 3 ], [ 6, 3046, 3 ], [ 6, 3047, 3 ], [ 6, 3052, 5 ],
  [ 6, 3053, 5 ], [ 6, 3054, 3 ], [ 6, 3061, 5 ], [ 6, 3066, 3 ] ]
gap> Length(KSfail51);
8
gap> KSfail51g:=List(KSfail51,x->CaratMatGroupZClass(x[1],x[2],x[3]));;
gap> List(KSfail51g,Order);
[ 120, 120, 120, 20, 20, 240, 40, 60 ]
gap> List(KSfail51g,StructureDescription);
[ "C2 x A5", "S5", "S5", "C5 : C4", "C5 : C4", "C2 x S5", "C2 x (C5 : C4)", "A5" ]
gap> last[6]; # the 6th group of the CARAT code [ 6, 3054, 3 ] is maximal
"C2 x S5"
gap> KSfail51gsub:=Set(ConjugacyClassesSubgroups2(KSfail51g[6]),
> x->CaratZClass(Representative(x)));;
gap> Difference(KSfail51,KSfail51gsub);
[ ]
gap> ips51f:=Flat(ips51);;
gap> ips51fc:=List(ips51f,CaratZClass);;
gap> KSfail51i:=List(KSfail51,x->Filtered([1..9939],y->ips51fc[y]=x));
[ [ 9630, 9633 ], [ 9635, 9638 ], [ 9642, 9645 ], [ 9648, 9651 ],
  [ 9653, 9656 ], [ 9660, 9665 ], [ 9670, 9675 ], [ 9679, 9681 ] ]
gap> KSfail51gg:=List(KSfail51i,x->[ips51f[x[1]],ips51f[x[2]]]);;
gap> List(KSfail51gg,x->List(x,y->List([1],[2..6]),
> z->CaratZClass(PartialMatrixGroup(y,z))));
[ [ [ [ 1, 2, 1 ], [ 5, 910, 3 ] ], [ [ 1, 2, 1 ], [ 5, 910, 4 ] ] ],
  [ [ [ 1, 1, 1 ], [ 5, 911, 3 ] ], [ [ 1, 1, 1 ], [ 5, 911, 4 ] ] ],
  [ [ [ 1, 2, 1 ], [ 5, 912, 3 ] ], [ [ 1, 2, 1 ], [ 5, 912, 4 ] ] ],
  [ [ [ 1, 2, 1 ], [ 5, 917, 3 ] ], [ [ 1, 2, 1 ], [ 5, 917, 4 ] ] ],
  [ [ [ 1, 1, 1 ], [ 5, 918, 3 ] ], [ [ 1, 1, 1 ], [ 5, 918, 4 ] ] ],
  [ [ [ 1, 2, 1 ], [ 5, 919, 3 ] ], [ [ 1, 2, 1 ], [ 5, 919, 4 ] ] ],
  [ [ [ 1, 2, 1 ], [ 5, 926, 5 ] ], [ [ 1, 2, 1 ], [ 5, 926, 6 ] ] ],
  [ [ [ 1, 1, 1 ], [ 5, 931, 3 ] ], [ [ 1, 1, 1 ], [ 5, 931, 4 ] ] ] ]
gap> List(last,y->List([y[1][1],y[1][2],y[2][1],y[2][2]],
> x->StructureDescription(CaratMatGroupZClass(x[1],x[2],x[3]))));
[ [ "C2", "C2 x A5", "C2", "C2 x A5" ],
  [ "1", "S5", "1", "S5" ],
  [ "C2", "S5", "C2", "S5" ],
  [ "C2", "C5 : C4", "C2", "C5 : C4" ],
  [ "1", "C5 : C4", "1", "C5 : C4" ],
  [ "C2", "C2 x S5", "C2", "C2 x S5" ],
  [ "C2", "C2 x (C5 : C4)", "C2", "C2 x (C5 : C4)" ],
  [ "1", "A5", "1", "A5" ] ]

```

## 5. GAP ALGORITHMS: THE FLABBY CLASS $[M_G]^{fl}$

Let  $G$  be a finite subgroup of  $GL(n, \mathbb{Z})$  and  $M = M_G$  be the  $G$ -lattice as in Definition 1.24. When  $G \simeq \text{Gal}(L/k)$  for a Galois extension  $L/k$ , the field  $L(M_G)^G$  may be regarded as the function field of an algebraic  $k$ -torus  $T$  which splits  $L$  via (1) (see Section 1).

In this section, we provide some GAP algorithms for computing the flabby class  $[M]^{fl}$  of  $M$  and for verifying whether  $[M]^{fl}$  is invertible and  $[M]^{fl} = 0$ . The following flow chart shows the structure of the GAP algorithms:



### 5.0. Determination whether $M_G$ is flabby (coflabby).

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. We provide GAP some algorithms for computing  $\hat{H}^{-1}(G, M_G)$  and  $H^1(G, M_G)$  and verifying whether the  $G$ -lattice  $M_G$  is flabby (resp. coflabby).

**Hminus1(G)** returns the Tate cohomology group  $\hat{H}^{-1}(G, M_G)$ .

**H1(G)** returns the cohomology group  $H^1(G, M_G)$ .

**IsFlabby(G)** returns whether  $G$ -lattice  $M_G$  is flabby or not.

**IsCoflabby(G)** returns whether  $G$ -lattice  $M_G$  is coflabby or not.

Note that in the algorithms below we use the formulas  $(\hat{B}^{-1}(G, M) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap M = \hat{Z}^{-1}(G, M)$ ,  $(\hat{B}^0(G, M) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap M = \hat{Z}^0(G, M)$  and  $(B^1(G, M) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap C^1(G, M) = Z^1(G, M)$  to compute  $\hat{H}^{-1}(G, M)$ ,  $\hat{H}^0(G, M)$  and  $H^1(G, M)$  respectively.

**Algorithm F0** (Determination whether  $M_G$  is flabby (coflabby)).

```

Hminus1:= function(g)
  local m,gg,i,s,r;
  m:=[];
  gg:=GeneratorsOfGroup(g);
  if gg=[] then
    return [];
  else
    for i in gg do
      m:=Concatenation(m,i-Identity(g));
    od;
    s:=SmithNormalFormIntegerMat(m);
    r:=Rank(s);
    return List([1..r],x->s[x][x]);
  fi;
end;

```

```

H1:= function(g)

```

```

local m,gg,i,s,r;
m:=[];
gg:=GeneratorsOfGroup(g);
if gg=[] then
  return [];
else
  for i in gg do
    m:=Concatenation(m,TransposedMat(i)-Identity(g));
  od;
  m:=TransposedMat(m);
  s:=SmithNormalFormIntegerMat(m);
  r:=Rank(s);
  return List([1..r],x->s[x][x]);
fi;
end;

IsFlabby:= function(g)
  local h;
  h:=List(ConjugacyClassesSubgroups2(g),Representative);
  return ForAll(h,x->Product(Hminus1(x))=1);
end;

IsCoflabby:= function(g)
  local h;
  h:=List(ConjugacyClassesSubgroups2(g),Representative);
  return ForAll(h,x->Product(H1(x))=1);
end;

```

### 5.1. Construction of the flabby class $[M_G]^{fl}$ of the $G$ -lattice $M_G$ .

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. We want to construct a flabby resolution  $0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$  with rank  $F$  not too large. If  $0 \rightarrow Q \rightarrow R \rightarrow M^\circ \rightarrow 0$  is a coflabby resolution of  $M^\circ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  where  $R$  is permutation and  $Q$  is coflasque, we can get a flasque resolution by taking a dual exact sequence  $0 \rightarrow M \rightarrow \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Z}) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}) \rightarrow 0$ .

We construct a coflabby resolution of  $M^\circ$  first. Let  $P^\circ$  be a permutation  $G$ -lattice and  $P^\circ \xrightarrow{f} M^\circ$  be a  $G$ -homomorphism. For a subgroup  $H$  of  $G$ , suppose that  $f$  maps  $(P^\circ)^H$  surjectively to  $(M^\circ)^H$ , i.e.

$$(2) \quad (P^\circ)^H \xrightarrow{f|_{(P^\circ)^H}} (M^\circ)^H \rightarrow 0 \text{ is exact.}$$

Then  $\hat{H}^0(H, P^\circ) \rightarrow \hat{H}^0(H, M^\circ)$  is surjective, so that  $\hat{H}^1(H, Q) = 0$  where  $Q = \mathrm{Ker} f$  by Lemma 2.5. In order to get a coflabby resolution of  $M^\circ$ , it is enough to construct a permutation  $G$ -lattice  $P^\circ$  and a  $G$ -homomorphism  $f$  such that (2) is satisfied for all subgroups  $H$  of  $G$ .

Let  $\mathcal{P}^\circ$  be a finite subset of  $M^\circ$  which is closed under the action of  $G$ . Let  $P^\circ$  be a free  $\mathbb{Z}$ -module over  $\mathcal{P}^\circ$ , i.e.  $P^\circ = \mathbb{Z}[\mathcal{P}^\circ]$ . The  $G$ -lattice  $P^\circ$  is permutation naturally, and for  $p \in \mathcal{P}^\circ$ ,  $P^\circ \ni p \mapsto p \in \mathcal{P}^\circ \subset M^\circ$  defines a  $G$ -homomorphism  $f : P^\circ \rightarrow M^\circ$ . If  $\mathcal{P}^\circ$  contains a  $\mathbb{Z}$ -basis of  $(M^\circ)^H$  for all subgroups  $H$  of  $G$ , then the condition (2) is satisfied for all subgroups  $H \subset G$ .

The condition (2) is consistent with the conjugation. Namely, (2) is satisfied for  $H$  if and only if it is satisfied for  $H^g = g^{-1}Hg$ . Thus it is enough to consider only the subgroups  $H$  not mutually conjugate.

Let  $\mathcal{H} = \{H_i\}_{i \in I}$  be the set of all conjugacy classes of subgroups of  $G$ . Let  $\mathcal{Q}_i$  be a free  $\mathbb{Z}$ -basis of  $(M^\circ)^{H_i}$ . Then

$$\mathcal{P}^\circ = \bigcup_{r \in \mathcal{R}} \mathrm{Orb}_G(r), \quad \mathcal{R} = \bigcup_{i \in I} \mathcal{Q}_i$$

provides  $P^\circ = \mathbb{Z}[\mathcal{P}^\circ]$  which satisfies (2) for all  $H_i \subset G$ . Therefore we may obtain a coflabby resolution of  $M$ :

$$(3) \quad 0 \rightarrow \mathrm{Ker} f \rightarrow P^\circ \xrightarrow{f} M^\circ \rightarrow 0.$$



The following algorithm (Algorithm F1) tries to find some “reduced” subset  $\mathcal{R}' \subset \mathcal{R}$  such that  $P^\circ = \mathbb{Z}[\mathcal{P}^\circ]$  satisfies the condition (2) where

$$(4) \quad \mathcal{P}^\circ = \bigcup_{r \in \mathcal{R}'} \text{Orb}_G(r).$$

This trial is performed in a computer calculation as follows:

The following algorithms construct a “reduced” flabby resolution of  $M_G$ :  $0 \rightarrow M_G \xrightarrow{\iota} P \xrightarrow{\phi} F \rightarrow 0$ .

`Z0lattice(G)` returns a  $\mathbb{Z}$ -basis of  $(M_G)^G$ .

`FindCoflabbyResolutionBase(G, HH)` returns a  $\mathbb{Z}$ -basis of a smaller permutation lattice  $P^\circ$  satisfying (2) for any  $H \in \mathcal{H}$ .

`FlabbyResolution(G)` returns a flabby resolution of  $M_G$  as follows:

`FlabbyResolution(G).actionP` returns the matrix representation of the action of  $G$  on  $P$ ;

`FlabbyResolution(G).actionF` returns the matrix representation of the action of  $G$  on  $F$ ;

`FlabbyResolution(G).injection` returns the matrix which corresponds to the injection  $\iota : M_G \rightarrow P$ ;

`FlabbyResolution(G).surjection` returns the matrix which corresponds to the surjection  $\phi : P \rightarrow F$ .

**Algorithm F1** (Construction of the flabby class  $[M_G]^{fl}$  of the  $G$ -lattice  $M_G$ ).

```

Z0lattice:= function(g)
  local gg,m,i;
  gg:=GeneratorsOfGroup(g);
  if gg=[] then
    return Identity(g);
  else
    m:=[];
    for i in gg do
      m:=Concatenation(m,TransposedMat(i)-Identity(g));
    od;
    m:=TransposedMat(m);
    return NullspaceIntMat(m);
  fi;
end;

ReduceCoflabbyResolutionBase:= function(g,hh,mi)
  local o,oo,z0,mi2;
  oo:=Orbits(g,mi);
  z0:=List(hh,Z0lattice);
  for o in oo do
    mi2:=Filtered(mi,x->(x in o)=fail);
    if ForAll([1..Length(hh)],i->LatticeBasis(List(Orbits(hh[i],mi2),Sum))=z0[i]) then
      mi:=mi2;
    fi;
  od;
  return mi;
end;

FindCoflabbyResolutionBase:= function(g,hh)
  local d,mi,h,z0,ll,i,o;
  d:=Length(Identity(g));
  mi:=[];
  for h in hh do
    z0:=Z0lattice(h);
    ll:=LatticeBasis(List(Orbits(h,mi),Sum));

```

```

    for i in z0 do
      if LatticeBasis(Concatenation(l1,[i]))<>l1 then
        o:=Orbit(g,i);
        mi:=Concatenation(mi,o);
        o:=List(Orbits(h,o),Sum);
        l1:=LatticeBasis(Concatenation(l1,o));
      fi;
    od;
  od;
  return ReduceCoflabbyResolutionBase(g,hh,mi);
end;

FlabbyResolution:= function(g)
  local tg,gg,d,th,mi,ms,o,r,gg1,gg2,v1,mg,img;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  o:=IdentityMat(r);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return rec(injection:=TransposedMat(mi),
              surjection:=NullMat(r,0),
              actionP:=TransposedMatrixGroup(Group(gg1,o))
            );
  else
    ms:=NullspaceIntMat(mi);
    v1:=NullspaceIntMat(TransposedMat(ms));
    mg:=Concatenation(v1,ms);
    img:=mg^-1;
    gg2:=List(gg1,x->mg*x*img);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    return rec(injection:=TransposedMat(mi),
              surjection:=TransposedMat(ms),
              actionP:=TransposedMatrixGroup(Group(gg1)),
              actionF:=TransposedMatrixGroup(Group(gg2))
            );
  fi;
end;

```

**Example 5.1** (Algorithm F1). The  $G$ -lattice  $M_G$  has the trivial flabby class  $[M_G]^{fl} = 0$  for  $G = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle \simeq C_3$ .

```
Read("FlabbyResolution.gap");
```

```

gap> G:=Group([[0,1],[-1,-1]]);
Group([ [ [ 0, 1 ], [ -1, -1 ] ] ])
gap> FlabbyResolution(G);
rec( injection := [ [ 1, 0, -1 ], [ 0, -1, 1 ] ],
     surjection := [ [ 1 ], [ 1 ], [ 1 ] ],
     actionP := Group([ [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ] ]),
     actionF := Group([ [ [ 1 ] ] ]) )
gap> FlabbyResolution(G).actionF; # F is trivial of rank 1

```

Group([ [ [ 1 ] ] ] ) )

## 5.2. Determination whether $[M_G]^{fl}$ is invertible.

Let  $G$  be a finite subgroup of  $GL(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. We provide the algorithm (Algorithm F2) for the determination whether  $[M_G]^{fl}$  is invertible. First, take a flabby resolution of  $M$ :

$$0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0.$$

If  $F$  is not coflabby, then  $[M]^{fl}$  is not invertible. If  $F$  is coflabby, then take a flabby resolution

$$(5) \quad 0 \rightarrow F \xrightarrow{\iota} Q \rightarrow E \rightarrow 0.$$

By Lemma 2.11, we find that  $[M]^{fl}$  is invertible  $\iff F$  is invertible  $\iff E$  is invertible  $\iff (5)$  splits. Thus it remains to check whether (5) splits, i.e. whether there exists  $\pi : Q \rightarrow F$  such that  $\pi \circ \iota = \text{id}_F$ .

We divide the standard basis of  $Q$  into  $G$ -orbits, and take a complete representative system  $\{x_\lambda\}$  of the  $G$ -orbits. Let  $H_\lambda$  be the stabilizer of  $x_\lambda$  in  $G$ . Then  $\pi(x_\lambda)$  in  $F^{H_\lambda}$ . Conversely if we fix an element of  $F^{H_\lambda}$  as  $\pi(x_\lambda)$ , then  $\pi$  is determined by the representatives  $\{x_\lambda\}$ .

We may have the necessary and the sufficient condition for  $\pi \circ \iota = \text{id}_M$ , and this becomes a system of linear equations. Hence this system of linear equations has an integer solution  $\iff$  the section  $\pi : Q \rightarrow F$  exists  $\iff [M]^{fl}$  is invertible. This computation is performed in GAP as follows:

IsInvertibleF(G) returns whether  $[M_G]^{fl}$  is invertible.

**Algorithm F2** (Determination whether  $[M_G]^{fl}$  is invertible).

```
IsInvertibleF:= function(g)
  local tg,gg,d,th,mi,mi2,ms,z0,l1,h,r,i,j,k,gg1,gg2,g1,g2,oo,iso,ker,
    tg2,th2,h2,l1,l2,v1,m1,m2;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return true;
  else
    ms:=NullspaceIntMat(mi);
    v1:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v1,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    for h in th2 do
      if Product(Hminus1(h))>1 then
        return false;
      fi;
    od;
    for h in th do
      if Product(Hminus1(h))>1 then
```

```

d:=r-d;
mi:=FindCoflabbyResolutionBase(tg2,th2);
r:=Length(mi);
gg1:=List(gg2,x->Permutation(x,mi));
gg2:=List(gg2,TransposedMat);
g2:=Group(gg2);
gg1:=List(gg1,x->x^-1);
g1:=Group(gg1);
mi:=TransposedMat(mi);
iso:=GroupHomomorphismByImagesNC(g1,g2,gg1,gg2);
oo:=OrbitsDomain(g1,[1..r]);
m1:=[];
for i in oo do
  h:=Stabilizer(g1,i[1]);
  l1:=List(RightCosetsNC(g1,h),Representative);
  l1:=List(l1,x->i[1]^x);
  l2:=List(l1,x->Image(iso,x));
  z0:=Z0lattice(Image(iso,h));
  for j in z0 do
    m2:=NullMat(r,d);
    for k in [1..Length(l1)] do
      m2[l1[k]]:=j*l2[k];
    od;
    Add(m1,Flat(mi*m2));
  od;
od;
m2:=Concatenation(m1,[Flat(IdentityMat(d))]);
m1:=NullspaceIntMat(m2);
return Gcd(TransposedMat(m1)[Length(m2)])=1;
fi;
od;
gg1:=List(gg,x->Permutation(x,mi));
gg:=GeneratorsOfGroup(g);
gg1:=List(gg1,x->x^-1);
g1:=Group(gg1);
mi:=TransposedMat(mi);
iso:=GroupHomomorphismByImagesNC(g1,g,gg1,gg);
oo:=OrbitsDomain(g1,[1..r]);
m1:=[];
for i in oo do
  h:=Stabilizer(g1,i[1]);
  l1:=List(RightCosetsNC(g1,h),Representative);
  l1:=List(l1,x->i[1]^x);
  l2:=List(l1,x->Image(iso,x));
  z0:=Z0lattice(Image(iso,h));
  for j in z0 do
    m2:=NullMat(r,d);
    for k in [1..Length(l1)] do
      m2[l1[k]]:=j*l2[k];
    od;
    Add(m1,Flat(mi*m2));
  od;
od;
m2:=Concatenation(m1,[Flat(IdentityMat(d))]);
m1:=NullspaceIntMat(m2);

```

```

    return Gcd(TransposedMat(m1)[Length(m2)])=1;
  fi;
end;

```

### 5.3. Computation of $E$ with $[[M_G]^{fl}]^{fl} = [E]$ .

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. By a result of Section 5.2 (Algorithm F2), we assume that  $[M]^{fl}$  is invertible. The next step is to determine whether  $[M]^{fl} = 0$ . First, we give a sufficient condition for  $[M]^{fl} = 0$ . Let  $0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0$  be a flabby resolution of  $M$ . By Algorithm F2, we can compute  $F$ , and by the assumption  $F$  is invertible. If  $F$  is still complicated, we take a flabby resolution of  $F$ :  $0 \rightarrow F \rightarrow Q \rightarrow E \rightarrow 0$ . Then  $E = 0 \implies F = 0 \implies [M]^{fl} = 0$ . By the same way, we define  $[M]^{fl^n} := [[M]^{fl^{n-1}}]^{fl}$  inductively. Then we obtain a sufficient condition for  $[M]^{fl} = 0$ , namely,  $[M]^{fl^n} = 0 \implies [M]^{fl} = 0$ . The following algorithm may compute  $E$  with  $[E] = [[M]^{fl}]^{fl}$  effectively.

`flfl(G)` returns the  $G$ -lattice  $E$  with  $[[M_G]^{fl}]^{fl} = [E]$ .

**Algorithm F3** (Computation of  $E$  with  $[[M_G]^{fl}]^{fl} = [E]$ ).

```

flfl:= function(g)
  local tg,gg,d,th,mi,ms,r,gg1,gg2,v1,mg,img,tg2,iso,ker;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return [];
  else
    ms:=NullspaceIntMat(mi);
    v1:=NullspaceIntMat(TransposedMat(ms));
    mg:=Concatenation(v1,ms);
    img:=mg^-1;
    gg2:=List(gg1,x->mg*x*img);
    gg2:=List(gg2,x->x[[d+1..r]][[d+1..r]]);
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th:=List(Filtered(th,x->IsSubset(x,ker)),x->Image(iso,x));
    tg:=tg2;
    gg:=gg2;
    d:=r-d;
    mi:=FindCoflabbyResolutionBase(tg,th);
    r:=Length(mi);
    gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
    if r=d then
      return [];
    else
      ms:=NullspaceIntMat(mi);
      v1:=NullspaceIntMat(TransposedMat(ms));
      mg:=Concatenation(v1,ms);
      img:=mg^-1;
      gg2:=List(gg1,x->mg*x*img);

```

```

      gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
      return TransposedMatrixGroup(Group(gg2));
    fi;
  fi;
end;

```

**Example 5.2** (Algorithm F3). Let  $\text{Imf}(n, i, j) \leq \text{GL}(n, \mathbb{Z})$  be the  $j$ -th  $\mathbb{Z}$ -class of the  $i$ -th  $\mathbb{Q}$ -class of the irreducible maximal finite group of dimension  $n$  (which corresponds to the GAP command `ImfMatrixGroup(n, i, j)`). Note that the maximal irreducible finite groups  $\text{Imf}(n, i, j)$  coincide with the maximal indecomposable finite groups  $\text{Indmf}(n, i, j)$  when  $n \leq 5$  (see Subsection 4.0).

By using the algorithm `flf1`, we may confirm that  $[M_G]^{fl} = 0$  for  $G = \text{Imf}(4, 4, 1)$  and  $\text{Imf}(5, 1, 1)$ . By Lemma 2.17, we obtain  $[M_H]^{fl} = 0$  for any subgroups  $H$  of  $G$ . There are 193 (resp. 953) conjugacy classes of subgroups of  $\text{Imf}(4, 4, 1)$  (resp.  $\text{Imf}(5, 1, 1)$ ).

```

Read("crystcat.gap");
Read("caratnumber.gap");
Read("FlabbyResolution.gap");

gap> G:=ImfMatrixGroup(4,4,1);
ImfMatrixGroup(4,4,1)
gap> Size(G);
384
gap> CrystCatZClass(G);
[ 4, 32, 21, 1 ]
gap> flf1(G);
[ ]
gap> Length(List(ConjugacyClassesSubgroups2(G),Representative)); # # of conjugacy classes
193

gap> G:=ImfMatrixGroup(5,1,1);
ImfMatrixGroup(5,1,1)
gap> Size(G);
3840
gap> CaratZClass(G);
[ 5, 942, 1 ]
gap> flf1(G);
[ ]
gap> Length(List(ConjugacyClassesSubgroups2(G),Representative)); # # of conjugacy classes
953

```

**Example 5.3** (Kunyavskii's birational classification of the algebraic  $k$ -tori of dimension 3 (Theorem 1.2)). By Voskresenskii's theorem (Theorem 1.1),  $[M_G]^{fl} = 0$  for all  $G$ -lattices  $M_G$  of rank  $\leq 2$ . Using the algorithm `flf1`, we may confirm Kunyavskii's theorem (Theorem 1.2). There exist 39 (resp. 34) decomposable (resp. indecomposable)  $G$ -lattices  $M_G$  of rank 3 (see Example 4.9).

By Voskresenskii's theorem and Lemma 2.14,  $[M_G]^{fl} = 0$  for 39 decomposable  $G$ -lattices  $M_G$ .

Using `flf1` and by Lemma 2.17, we may see that  $[M_G]^{fl} = 0$  for any subgroups  $G$  of  $\text{Imf}(3, 1, 1) \simeq C_2 \times S_4$ , the group  $G_1 \simeq D_4$  of the GAP code (3, 4, 5, 2) and the group  $G_2 \simeq S_4$  of the GAP code (3, 7, 4, 3). Namely,  $L(M_G)^G$  is stably  $k$ -rational. Note that Kunyavskii's theorem claims not only the stably  $k$ -rationality but also the  $k$ -rationality, and we could not confirm the  $k$ -rationality by this method.

There exist exactly 15 groups which are not subgroups of  $\text{Imf}(3, 1, 1)$ ,  $G_1$  and  $G_2$  (see Table 2 in Theorem 1.1). Indeed, using the algorithm `IsInvertibleF`, we may confirm that  $[M_G]^{fl}$  is not invertible for all the 15 groups  $G$ . Namely,  $L(M_G)^G$  is not retract  $k$ -rational.

```

Read("crystcat.gap");
Read("FlabbyResolution.gap");

gap> G:=ImfMatrixGroup(3,1,1); # G=C2xS4
ImfMatrixGroup(3,1,1)
gap> CrystCatZClass(G);
[ 3, 7, 5, 1 ]
gap> flfl(G);
[ ]
gap> flfl(MatGroupZClass(3,4,5,2)); # G=D4
[ ]
gap> flfl(MatGroupZClass(3,7,4,3)); # G=S4
[ ]

gap> ld3:=LatticeDecompositions(3);
gap> Partitions(3);
[ [ 1, 1, 1 ], [ 2, 1 ], [ 3 ] ]
gap> List(ld3,Length);
[ 8, 31, 34 ]
gap> ind3:=ld3[3];
gap> imf311sub:=Set(ConjugacyClassesSubgroups2(G),
> x->CrystCatZClass(Representative(x)));
gap> G3452sub:=Set(ConjugacyClassesSubgroups2(MatGroupZClass(3,4,5,2)),
> x->CrystCatZClass(Representative(x)));
gap> G3743sub:=Set(ConjugacyClassesSubgroups2(MatGroupZClass(3,7,4,3)),
> x->CrystCatZClass(Representative(x)));
gap> N3:=Difference(ind3,Union(imf311sub,G3452sub,G3743sub));
[ [ 3, 3, 1, 3 ], [ 3, 3, 3, 3 ], [ 3, 3, 3, 4 ], [ 3, 4, 3, 2 ],
  [ 3, 4, 4, 2 ], [ 3, 4, 6, 3 ], [ 3, 4, 7, 2 ], [ 3, 7, 1, 2 ],
  [ 3, 7, 2, 2 ], [ 3, 7, 2, 3 ], [ 3, 7, 3, 2 ], [ 3, 7, 3, 3 ],
  [ 3, 7, 4, 2 ], [ 3, 7, 5, 2 ], [ 3, 7, 5, 3 ] ]
gap> Length(N3);
15
gap> Filtered(N3,x->IsInvertibleF(MatGroupZClass(x[1],x[2],x[3],x[4]))=true);
[ ]

```

#### 5.4. Possibility for $[M_G]^{fl} = 0$ .

We will devise an algorithm which, in a favorable situation, will enable us to show that some  $G$ -lattices are not stably permutation.

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. By a result of Section 5.2 (Algorithm F2), we assume that  $[M]^{fl}$  is invertible.

Each isomorphism class of irreducible permutation  $G$ -lattices corresponds to a conjugacy class of subgroup  $H$  of  $G$  by  $H \leftrightarrow \mathbb{Z}[G/H]$ . Let  $H_1, \dots, H_r$  be conjugacy classes of subgroups of  $G$  whose ordering corresponds to the GAP function `ConjugacyClassesSubgroups2(G)` (see Section 4). Let  $F$  be the flabby class of  $M_G$ .

We assume that  $F$  is stably permutation, i.e. for  $x_{r+1} = \pm 1$ ,

$$\left( \bigoplus_{i=1}^r x_i \mathbb{Z}[G/H_i] \right) \oplus x_{r+1} F \simeq \bigoplus_{i=1}^r y_i \mathbb{Z}[G/H_i].$$

Define  $a_i = x_i - y_i$  and  $b_1 = x_{r+1}$ . Then we have for  $b_1 = \pm 1$ ,

$$(6) \quad \bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i] \simeq -b_1 F.$$

By computing some  $\mathbb{Z}$ -class invariants, we will give a necessary condition for  $[M_G]^{fl} = 0$ .

Let  $\{c_1, \dots, c_r\}$  be a set of complete representatives of the conjugacy classes of  $G$ . Let  $A_i(c_j)$  be the matrix representation of the factor coset action of  $c_j \in G$  on  $\mathbb{Z}[G/H_i]$  and  $B(c_j)$  be the matrix representation of the action of  $c_j \in G$  on  $F$ . By (6), for each  $c_j \in G$ , we have

$$(7) \quad \sum_{i=1}^r a_i \operatorname{tr} A_i(c_j) + b_1 \operatorname{tr} B(c_j) = 0$$

where  $\operatorname{tr} A$  is the trace of the matrix  $A$ . Similarly, we consider the rank of  $H^0 = \widehat{Z}^0$ . For each  $H_j$ , we get

$$(8) \quad \sum_{i=1}^r a_i \operatorname{rank} \widehat{Z}^0(H_j, \mathbb{Z}[G/H_i]) + b_1 \operatorname{rank} \widehat{Z}^0(H_j, F) = 0.$$

Finally, we compute  $\widehat{H}^0$ . Let  $Sy_p(A)$  be a  $p$ -Sylow subgroup of an abelian group  $A$ .  $Sy_p(A)$  can be written as a direct product of cyclic groups uniquely. Let  $n_{p,e}(Sy_p(A))$  be the number of direct summands of cyclic groups of order  $p^e$ . For each  $H_j, p, e$ , we get

$$(9) \quad \sum_{i=1}^r a_i n_{p,e}(Sy_p(\widehat{H}^0(H_j, \mathbb{Z}[G/H_i]))) + b_1 n_{p,e}(Sy_p(\widehat{H}^0(H_j, F))) = 0.$$

By the equalities (7), (8) and (9), we may get a system of linear equations in  $a_1, \dots, a_r, b_1$  over  $\mathbb{Z}$ . Namely, we have that  $[M_G]^{fl} = 0 \implies$  the exist  $a_1, \dots, a_r \in \mathbb{Z}$  and  $b_1 = \pm 1$  which satisfy (6)  $\implies$  this system of linear equations has a integer solution in  $a_1, \dots, a_r$  with  $b_1 = \pm 1$ . In particular, if this system of linear equations has no integer solutions, then we conclude that  $[M_G]^{fl} \neq 0$ .

$H0(G)$  returns the Tate cohomology group  $\widehat{H}^0(G, M_G)$ .

**PossibilityOfStablyPermutationF(G)** returns a basis  $\mathcal{L} = \{l_1, \dots, l_s\}$  of the solution space of the system of linear equations which is obtained by the equalities (7), (8) and (9).

**PossibilityOfStablyPermutationM(G)** returns the same as **PossibilityOfStablyPermutationF(G)** but with respect to  $M_G$  instead of  $F$ . (We will use this in Section 6.)

**Algorithm F4** (Possibility for  $[M_G]^{fl} = 0$ ).

```

H0:= function(g)
  local m,s,r;
  m:=Sum(g);
  s:=SmithNormalFormIntegerMat(m);
  r:=Rank(s);
  return List([1..r], x->s[x][x]);
end;

PossibilityOfStablyPermutationH:= function(g,hh)
  local gg,hg,hgg,hom,c,h,m,m1,m2,v,h0,og,oh,p,e;
  gg:=GeneratorsOfGroup(g);
  hg:=List(hh, x->RightCosets(g,x));
  hgg:=List(hg, x->List(gg, y->Permutation(y,x, OnRight)));
  hom:=List(hgg, x->GroupHomomorphismByImages(g, Group(x,()), gg, x));
  c:=List(ConjugacyClasses(g), Representative);
  og:=Order(g);
  m:=List(c, x->List([1..Length(hh)], y->og/Order(hh[y])-NrMovedPoints(Image(hom[y], x))));
  v:=List(c, Trace);
  for h in hh do
    h0:=H0(h);
    oh:=Order(h);
    m1:=List([1..Length(hh)],
      x->List(OrbitLengths(Image(hom[x], h), [1..og/Order(hh[x])]), y->oh/y));
    Add(m, List(m1, Length));
    Add(v, Length(h0));
    if oh>1 then

```



```

    for p in Set(FactorsInt(oh)) do
      for e in [1..PadicValuation(oh,p)] do
        Add(m,List(m1,x->Number(x,y->PadicValuation(y,p)=e)));
        Add(v,Number(h0,x->PadicValuation(x,p)=e));
      od;
    od;
  fi;
od;
m:=TransposedMat(m);
m:=Concatenation(m,[v]);
return NullspaceIntMat(m);
end;

```

```

PossibilityOfStablyPermutationF:= function(g)
  local tg,gg,d,th,mi,ms,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return [1,-1];
  else
    ms:=NullspaceIntMat(mi);
    v:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    g2:=TransposedMatrixGroup(tg2);
    h2:=List(th2,TransposedMatrixGroup);
    return PossibilityOfStablyPermutationH(g2,h2);
  fi;
end;

```

```

PossibilityOfStablyPermutationM:= function(g)
  local hh;
  hh:=List(ConjugacyClassesSubgroups2(g),Representative);
  return PossibilityOfStablyPermutationH(g,hh);
end;

```

**Example 5.4** (Algorithm F4: Possibility for  $[M_G]^{fl} = 0$ ). Let  $G \simeq F_{20}$  be the Frobenius group of order 20 of the GAP code (4, 31, 1, 3) or (4, 31, 1, 4) as in Table 2. By Algorithm F2, we may check that  $[M_G]^{fl}$  is invertible. We will show that  $[M_G]^{fl} \neq 0$ . There exist 6 conjugacy classes of subgroups  $\{1\}$ ,  $H_2 \simeq C_2$ ,  $H_3 \simeq C_4$ ,  $H_4 \simeq C_5$ ,  $H_5 \simeq D_5$  and  $G$  of order 1, 2, 4, 5, 10 and 20 respectively. The corresponding permutation  $G$ -lattices are  $\mathbb{Z}[G]$ ,  $\mathbb{Z}[G/H_2]$ ,  $\mathbb{Z}[G/H_3]$ ,  $\mathbb{Z}[G/H_4]$ ,  $\mathbb{Z}[G/H_5]$  and  $\mathbb{Z}$  of rank 20, 10, 5, 4, 2 and 1 (this ordering is determined by the GAP function `ConjugacyClassesSubgroups2(G)` (see Section 4)). By Algorithm F1, we may obtain the flabby  $G$ -lattice  $F$  of rank 16 via the function `FlabbyResolution(G).actionF` where  $0 \rightarrow M_G \rightarrow P \rightarrow F$  is a flabby resolution of  $M_G$ .

Let  $\mathcal{L} = \{l_1, \dots, l_s\}$  be a list of lists obtained by the GAP function `PossibilityOfStablyPermutationF(G)`. Put  $\mathcal{U} = \mathbb{Z}l_1 + \dots + \mathbb{Z}l_s$  i.e. the set of all linear combinations of  $\mathcal{L}$  over  $\mathbb{Z}$ . When  $[a_1, \dots, a_r, b_1] \in \mathcal{U}$ , there is a possibility that

$$\bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i] \simeq -b_1 F.$$

By  $\mathcal{U} = \langle [1, 1, 0, 1, -1, 0, -2] \rangle$ , there is a possibility that

$$\mathbb{Z}[G] \oplus \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_4] \simeq \mathbb{Z}[G/H_5] \oplus 2F.$$

However,  $b_1$  should be  $\pm 1$ . This implies that  $F$  is not stably permutation, hence  $[M_G]^{fl} \neq 0$ . In other words,  $L(M_G)^G$  is not stably but retract  $k$ -rational.

By  $H^2(G, \mathbb{Z}[G]) = 0$ ,  $H^2(G, \mathbb{Z}[G/H_2]) = \mathbb{Z}/2\mathbb{Z}$  and  $H^2(G, \mathbb{Z}[G/H_4]) = \mathbb{Z}/5\mathbb{Z}$  while  $H^2(G, \mathbb{Z}[G/H_5]) = \mathbb{Z}/2\mathbb{Z}$  and  $H^2(G, F) = 0$ , we also see that  $kF$  is not stably permutation for all  $k \geq 1$ .

```
Read("FlabbyResolution.gap");
```

```
gap> G:=MatGroupZClass(4,31,1,3);; # G=F20
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 16
16
gap> IsInvertibleF(G);
true
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C4", "C5", "D10", "C5 : C4" ]
gap> PossibilityOfStablyPermutationF(G); # checking [M]^{fl}: non-zero
[ [ 1, 1, 0, 1, -1, 0, -2 ] ]

gap> G:=MatGroupZClass(4,31,1,4);; # G=F20
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 16
16
gap> IsInvertibleF(G);
true
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C4", "C5", "D10", "C5 : C4" ]
gap> PossibilityOfStablyPermutationF(G); # checking [M]^{fl}: non-zero
[ [ 1, 1, 0, 1, -1, 0, -2 ] ]
```

### 5.5. Verification of $[M_G]^{fl} = 0$ : Method I.

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. Next we try to check whether the possibility of the isomorphism as in (6) actually holds. The following algorithm tries to find the isomorphism using the GAP function `RepresentativeAction(GL(n, Integers), G1, G2)`.

`Nlist(l)` returns the negative part of the list  $l$ .

`Plist(l)` returns the positive part of the list  $l$ .

`StablyPermutationFCheck(G, L1, L2)` returns the matrix  $P$  which satisfies  $G_1 P = P G_2$  where  $G_1$  (resp.  $G_2$ ) is the matrix representation group of the action of  $G$  on  $(\bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i]) \oplus b_1 F$  (resp.  $(\bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i]) \oplus b'_1 F$ ) with the isomorphism

$$(10) \quad \left( \bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i] \right) \oplus b_1 F \simeq \left( \bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i] \right) \oplus b'_1 F$$

for lists  $L_1 = [a_1, \dots, a_r, b_1]$  and  $L_2 = [a'_1, \dots, a'_r, b'_1]$ , if  $P$  exists. If such  $P$  does not exist, this returns **false**.

`StablyPermutationMCheck(G, L1, L2)` returns the same as `StablyPermutationFCheck(G, L1, L2)` but with respect to  $M_G$  instead of  $F$ . (We will use this in Section 6.)

If the rank of  $F$  is small enough, `StablyPermutationFCheck` works well. However, if the rank of  $F$  is not small, `StablyPermutationFCheck` does not return the answer in a suitable time, and then we need to make more efforts. We will explain this in the next section (Section 5.6).

**Algorithm F5** (Verification of  $[M_G]^{fl} = 0$ : Method I).

(The following algorithm needs the *CARAT* package, in particular for the command `RepresentativeAction`.)

```

Nlist:= function(l)
  return List(l,x->Maximum([-x,0]));
end;

Plist:= function(l)
  return List(l,x->Maximum([x,0]));
end;

CosetRepresentation:= function(g,h)
  local gg,hg,og;
  gg:=GeneratorsOfGroup(g);
  hg:=SortedList(RightCosets(g,h));
  og:=Length(hg);
  return List(gg,x->PermutationMat(Permutation(x,hg,OnRight),og));
end;

StablyPermutationCheckH:= function(g,hh,c1,c2)
  local gg,g1,g2,dx,m,i,j,d;
  gg:=List(hh,x->CosetRepresentation(g,x));
  Add(gg,GeneratorsOfGroup(g));
  g1:=[];
  g2:=[];
  for i in [1..Length(gg[1])] do
    m:=[];
    for j in [1..Length(gg)] do
      m:=Concatenation(m,List([1..c1[j]],x->gg[j][i]));
    od;
    Add(g1,DirectSumMat(m));
    m:=[];
    for j in [1..Length(gg)] do
      m:=Concatenation(m,List([1..c2[j]],x->gg[j][i]));
    od;
    Add(g2,DirectSumMat(m));
  od;
  d:=Length(g1[1]);
  if d<>Length(g2[1]) then
    return fail;
  else
    return RepresentativeAction(GL(d,Integers),Group(g1),Group(g2));
  fi;
end;

StablyPermutationMCheck:= function(g,c1,c2)
  local h;
  h:=List(ConjugacyClassesSubgroups2(g),Representative);
  return StablyPermutationCheckH(g,h,c1,c2);
end;

```

```

StablyPermutationFCheck:= function(g,c1,c2)
  local tg,gg,d,th,mi,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return true;
  else
    ms:=NullspaceIntMat(mi);
    v:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    g2:=TransposedMatrixGroup(tg2);
    h2:=List(th2,TransposedMatrixGroup);
    return StablyPermutationCheckH(g2,h2,c1,c2);
  fi;
end;

```

**Example 5.5** (Algorithm F5: Method I (1)). Let  $G = \text{Imf}(4, 3, 1) \simeq S_5 \times C_2$  be the group of order 240 of the GAP code (4, 31, 7, 1). By Algorithm F1, the rank of the flabby class  $F$  of  $G$  is 6. By Algorithm F5, the following possibility of the isomorphism exists:  $\mathbb{Z}[G/H_{52}] \oplus \mathbb{Z}[G/H_{54}] \simeq \mathbb{Z} \oplus F$  on which the matrix representation groups of the action of  $G$  both sides are  $G_1$  and  $G_2$  respectively where  $H_{52} \simeq S_4 \times C_2$  and  $H_{54} \simeq S_5$ . We may confirm the isomorphism via `StablyPermutationFCheck(G, Nlist(1), Plist(1))` which returns the matrix  $P$  with  $G_1 P = P G_2$ . This implies  $[M_G]^{f_l} = 0$ , and hence  $L(M_G)^G$  is stably  $k$ -rational.

```

Read("crystcat.gap");
Read("FlabbyResolution.gap");

gap> G:=ImfMatrixGroup(4,3,1); # G=C2xS5
ImfMatrixGroup(4,3,1)
gap> CrystCatZClass(G);
[ 4, 31, 7, 1 ]
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 6
6
gap> ll:=PossibilityOfStablyPermutationF(G);
gap> Length(ll);
18
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, -1, -1 ]
gap> Length(l);
58
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C2", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C2 x C2", "C4",

```

```

gap> G:=CaratMatGroupZClass(5,946,2); # G=S5
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 5
5
gap> CaratZClass(FlabbyResolution(G).actionF);
[ 5, 911, 4 ]
gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 0, 0, -1, 0, 0, -4, 0, -2, 1, 2, 0, -1, -1, 0, 4, 0, 1, -4, 4 ],
  [ 0, 1, 0, 0, 0, -1, -1, 0, -1, 0, 0, 0, 0, 0, 1, 1, 0, 0, -1, 1 ],
  [ 0, 0, 1, 0, 0, 0, -2, 0, -1, 0, 1, 0, -1, -1, 0, 2, 0, 1, -2, 2 ],
  [ 0, 0, 0, 0, 1, 2, -2, 0, -2, 1, 2, -2, -1, -2, -2, 2, 0, 1, -2, 4 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1 ]
gap> Length(l);
20
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Repr
[ "1", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C4", "C5", "C6",
  "S3", "S3", "D8", "D10", "A4", "D12", "C5 : C4", "S4", "A5", "S5" ]
gap> ss[17];
"S4"
gap> StablyPermutationFCheck(G,Nlist(l),Plist(l));
fail
gap> l2:=IdentityMat(Length(l))[Length(l)-1];

```

```
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ]
gap> StablyPermutationFCheck(G,Nlist(1)+12,Plist(1)+12);
[ [ 2, 2, 2, 2, 2, 3 ],
  [ 0, 0, -1, -1, -1, -1 ],
  [ -1, -1, -1, 0, 0, -1 ],
  [ -1, 0, 0, -1, -1, -1 ],
  [ 0, -1, -1, 0, -1, -1 ],
  [ 0, -1, 0, -1, -1, -1 ] ]
```

### 5.6. Verification of $[M_G]^{fl} = 0$ : Method II.

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. The function `RepresentativeAction( $\mathrm{GL}(n, \text{Integers})$ ,  $G_1, G_2$ )` in Algorithm F5 may not work well when  $n$  is not small. We will provide another method in order to confirm the isomorphism (10), i.e.

$$(10) \quad \left( \bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i] \right) \oplus b_1 F \simeq \left( \bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i] \right) \oplus b'_1 F,$$

although it is needed by trial and error.

Our aim is to find the matrix  $P$  which satisfies  $G_1 P = P G_2$  rapidly. If we can choose a matrix with determinant  $\det P = \pm 1$ ,  $G_1$  and  $G_2$  are  $\mathrm{GL}(n, \mathbb{Z})$ -conjugate, and hence the isomorphism (10) established. This implies that the flabby class  $[M_G]^{fl} = 0$ .

`StablyPermutationFCheckP(G,L1,L2)` returns a basis  $\mathcal{P} = \{P_1, \dots, P_m\}$  of the solution space of  $G_1 P = P G_2$  where  $G_1$  (resp.  $G_2$ ) is the matrix representation group of the action of  $G$  on  $(\bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i]) \oplus b_1 F$  (resp.  $(\bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i]) \oplus b'_1 F$ ) with the isomorphism (10) for lists  $L_1 = [a_1, \dots, a_r, b_1]$  and  $L_2 = [a'_1, \dots, a'_r, b'_1]$ , if  $P$  exists. If such  $P$  does not exist, this returns `[ ]`.

`StablyPermutationMCheckP(G,L1,L2)` returns the same as `StablyPermutationFCheckP(G,L1,L2)` but with respect to  $M_G$  instead of  $F$ . (We will use these in Section 6.)

`StablyPermutationFCheckMat(G,L1,L2,P)` returns `true` if  $G_1 P = P G_2$  and  $\det P = \pm 1$  where  $G_1$  (resp.  $G_2$ ) is the matrix representation group of the action of  $G$  on  $(\bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i]) \oplus b_1 F$  (resp.  $(\bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i]) \oplus b'_1 F$ ) with the isomorphism (10) for lists  $L_1 = [a_1, \dots, a_r, b_1]$  and  $L_2 = [a'_1, \dots, a'_r, b'_1]$ . If not, this returns `false`.

`StablyPermutationMCheckMat(G,L1,L2,P)` returns the same as `StablyPermutationFCheckMat(G,L1,L2,P)` but with respect to  $M_G$  instead of  $F$ . (We will use these in Section 6.)

`StablyPermutationFCheckGen(G,L1,L2)` returns the list  $[\mathcal{M}_1, \mathcal{M}_2]$  where  $\mathcal{M}_1 = [g_1, \dots, g_t]$  (resp.  $\mathcal{M}_2 = [g'_1, \dots, g'_t]$ ) is a list of the generators of  $G_1$  (resp.  $G_2$ ) which is the matrix representation group of the action of  $G$  on  $(\bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i]) \oplus b_1 F$  (resp.  $(\bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i]) \oplus b'_1 F$ ) with the isomorphism (10) for lists  $L_1 = [a_1, \dots, a_r, b_1]$  and  $L_2 = [a'_1, \dots, a'_r, b'_1]$ .

`StablyPermutationMCheckGen(G,L1,L2)` returns the same as `StablyPermutationFCheckGen(G,L1,L2)` but with respect to  $M_G$  instead of  $F$ . (We will use these in Section 6.)

### Algorithm F6 (Verification of $[M_G]^{fl} = 0$ : Method II).

```
TransformationMat:= function(l1,l2)
  local d1,d2,l,m,p,i,j;
  d1:=Length(l1[1]);
  d2:=Length(l2[1]);
  l:=Length(l1);
  m:=[];
  for i in [1..d1] do
    for j in [1..d2] do
      p:=NullMat(d1,d2);
      p[i][j]:=1;
```

```

        Add(m,Flat(List([1..1],x->l1[x]*p-p*l2[x])));
    od;
od;
p:=NullspaceIntMat(m);
return List(p,x->List([1..d1],y->x{[(y-1)*d2+1..y*d2]}));
end;

StablyPermutationCheckHP:= function(g,hh,c1,c2)
    local gg,l,m,m1,m2,tm,d1,d2,s1,s2,i,j,k;
    gg:=List(hh,x->CosetRepresentation(g,x));
    Add(gg,GeneratorsOfGroup(g));
    l:=List([1..Length(gg)],x->Length(gg[x][1]));
    d1:=Sum([1..Length(gg)],x->c1[x]*l[x]);
    d2:=Sum([1..Length(gg)],x->c2[x]*l[x]);
    m:=[];
    s1:=0;
    for i in [1..Length(gg)] do
        if c1[i]>0 then
            m1:=[];
            s2:=0;
            for j in [1..Length(gg)] do
                if c2[j]>0 then
                    tm:=TransformationMat(gg[i],gg[j]);
                    for k in [1..c2[j]] do
                        m2:=List(tm,x->TransposedMat(Concatenation(
                            [NullMat(s2,l[i]),TransposedMat(x),
                                NullMat(d2-s2-l[j],l[i])])));
                        m1:=Concatenation(m1,m2);
                        s2:=s2+l[j];
                    od;
                fi;
            od;
            m1:=LatticeBasis(List(m1,Flat));
            m1:=List(m1,x->List([1..l[i]],y->x{[(y-1)*s2+1..y*s2]}));
            for k in [1..c1[i]] do
                m:=Concatenation(m,List(m1,x->Concatenation(
                    [NullMat(s1,d2),x,NullMat(d1-s1-l[i],d2)]));
                s1:=s1+l[i];
            od;
        fi;
    od;
    return m;
end;

StablyPermutationMCheckP:= function(g,c1,c2)
    local h;
    h:=List(ConjugacyClassesSubgroups2(g),Representative);
    return StablyPermutationCheckHP(g,h,c1,c2);
end;

StablyPermutationFCheckP:= function(g,c1,c2)
    local tg,gg,d,th,mi,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
    tg:=TransposedMatrixGroup(g);
    gg:=GeneratorsOfGroup(tg);
    d:=Length(Identity(g));

```

```

th:=List(ConjugacyClassesSubgroups2(tg),Representative);
mi:=FindCoflabbyResolutionBase(tg,th);
r:=Length(mi);
gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
if r=d then
    return true;
else
    ms:=NullspaceIntMat(mi);
    v:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    g2:=TransposedMatrixGroup(tg2);
    h2:=List(th2,TransposedMatrixGroup);
    return StablyPermutationCheckHP(g2,h2,c1,c2);
fi;
end;

StablyPermutationCheckHMat:= function(g,hh,c1,c2,p)
    local gg,g1,g2,dx,m,i,j,d;
    gg:=List(hh,x->CosetRepresentation(g,x));
    Add(gg,GeneratorsOfGroup(g));
    g1:=[];
    g2:=[];
    for i in [1..Length(gg[1])] do
        m:=[];
        for j in [1..Length(gg)] do
            m:=Concatenation(m,List([1..c1[j]],x->gg[j][i]));
        od;
        Add(g1,DirectSumMat(m));
        m:=[];
        for j in [1..Length(gg)] do
            m:=Concatenation(m,List([1..c2[j]],x->gg[j][i]));
        od;
        Add(g2,DirectSumMat(m));
    od;
    d:=Length(g1[1]);
    if d<>Length(g2[1]) or d<>Length(p) or DeterminantMat(p)^2<>1 then
        return fail;
    else
        return List(g1,x->x^p)=g2;
    fi;
end;

StablyPermutationMCheckMat:= function(g,c1,c2,p)
    local h;
    h:=List(ConjugacyClassesSubgroups2(g),Representative);
    return StablyPermutationCheckHMat(g,h,c1,c2,p);
end;

```



```

StablyPermutationFCheckMat:= function(g,c1,c2,p)
  local tg,gg,d,th,mi,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return true;
  else
    ms:=NullspaceIntMat(mi);
    v:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    g2:=TransposedMatrixGroup(tg2);
    h2:=List(th2,TransposedMatrixGroup);
    return StablyPermutationCheckHMat(g2,h2,c1,c2,p);
  fi;
end;

```

```

StablyPermutationCheckHGen:= function(g,hh,c1,c2)
  local gg,g1,g2,dx,m,i,j,d;
  gg:=List(hh,x->CosetRepresentation(g,x));
  Add(gg,GeneratorsOfGroup(g));
  g1:=[];
  g2:=[];
  for i in [1..Length(gg[1])] do
    m:=[];
    for j in [1..Length(gg)] do
      m:=Concatenation(m,List([1..c1[j]],x->gg[j][i]));
    od;
    Add(g1,DirectSumMat(m));
    m:=[];
    for j in [1..Length(gg)] do
      m:=Concatenation(m,List([1..c2[j]],x->gg[j][i]));
    od;
    Add(g2,DirectSumMat(m));
  od;
  return [g1,g2];
  d:=Length(g1[1]);
end;

```

```

StablyPermutationMCheckGen:= function(g,c1,c2)
  local h;
  h:=List(ConjugacyClassesSubgroups2(g),Representative);
  return StablyPermutationCheckHGen(g,h,c1,c2);
end;

```

```

StablyPermutationFCheckGen:= function(g,c1,c2)
  local tg,gg,d,th,mi,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  mi:=FindCoflabbyResolutionBase(tg,th);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return true;
  else
    ms:=NullspaceIntMat(mi);
    v:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    g2:=TransposedMatrixGroup(tg2);
    h2:=List(th2,TransposedMatrixGroup);
    return StablyPermutationCheckHGen(g2,h2,c1,c2);
  fi;
end;

```

**Example 5.7** (Algorithm F6: Method II). Let  $M_G$  be the  $G$ -lattice where  $G = \text{Imf}(4, 2, 1) \simeq D_6^2 \rtimes C_2$  is the group of order 288 of the GAP code (4, 30, 13, 1). We will show that  $[M]^{fl} = [F] = 0$ . Indeed, we can verify that  $F$  is of rank 8 and

$$\mathbb{Z}[G/H_{196}] \oplus \mathbb{Z}[G/H_{212}] \simeq F \oplus \mathbb{Z}[G/H_{217}]$$

where  $H_{196} \simeq C_2^2 \times D_6$ ,  $H_{212} \simeq C_2 \times S_3^2$  and  $H_{217} \simeq D_6^2$  (the rank of the both sides is  $6 + 4 = 8 + 2 = 10$ ). By comparing with Algorithm F5: Method I, we may obtain the matrix representations of  $G_1$  and  $G_2$  which corresponds to the action of  $G$  on  $\mathbb{Z}[G/H_{196}] \oplus \mathbb{Z}[G/H_{212}]$  and  $F \oplus \mathbb{Z}[G/H_{217}]$  respectively and the matrix  $P$  which satisfies  $\sigma_1 P = P \sigma_2$  for any  $\sigma_1 \in G_1$  and  $\sigma_2 \in G_2$ .

```

gap> Read("crystcat.gap");
gap> Read("FlabbyResolution.gap");

gap> G:=ImfMatrixGroup(4,2,1);
ImfMatrixGroup(4,2,1)
gap> StructureDescription(G);
"(C2 x C2 x S3 x S3) : C2"
gap> CrystCatZClass(G);
[ 4, 30, 13, 1 ]
gap> GeneratorsOfGroup(FlabbyResolution(G).actionF); # F is of rank 8
[ [ [ 0, 1, 0, 0, 0, 0, 0, 0 ], [ 1, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 1, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 1, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1 ] ],
  [ [ 0, 0, 0, 0, 0, 1, 0, 0 ], [ 1, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 0, 0, 0, 0 ],
    [ 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 1, 0, 0, 0 ], [ 0, 1, 0, 1, 0, -1, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 1, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1 ] ],

```

```
[ [ 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0, 0 ], [ 1, 0, 0, 0, 0, 0, 0, 0 ],  
    [ 0, 0, 0, 0, 0, 0, 1, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1 ],  
    [ 0, 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0, 0 ] ] ]  
gap> ll:=PossibilityOfStablyPermutationF(G);;  
gap> l:=ll[Length(ll)];  
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
  0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]  
gap> Length(l);  
225  
gap> [l[196],l[212],l[217],l[225]];  
[ 1, 1, -1, -1 ]  
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));;  
gap> [ss[196],ss[212],ss[217]];  
[ "C2 x C2 x C2 x S3", "(S3 x S3) : C2", "C2 x C2 x S3 x S3" ]  
gap> bp:=StablyPermutationFCheckP(G,Nlist(l),Plist(l));;  
gap> Length(bp);  
10  
gap> Length(bp[1]); # rank of the both sides of (10) is 10  
10  
gap> cc:=Filtered(Tuples([0,1],10),x->Determinant(x*bp)^2=1);  
[ [ 0, 1, 1, 0, 0, 0, 1, 0, 1, 0 ], [ 0, 1, 1, 0, 0, 0, 1, 1, 0, 0 ],  
  [ 0, 1, 1, 0, 0, 1, 0, 0, 1, 0 ], [ 0, 1, 1, 0, 0, 1, 0, 1, 0, 0 ],  
  [ 0, 1, 1, 0, 1, 0, 1, 0, 1, 1 ], [ 0, 1, 1, 0, 1, 0, 1, 1, 0, 1 ],  
  [ 0, 1, 1, 0, 1, 1, 0, 0, 1, 1 ], [ 0, 1, 1, 0, 1, 1, 0, 1, 0, 1 ],  
  [ 1, 0, 0, 1, 0, 0, 1, 0, 1, 0 ], [ 1, 0, 0, 1, 0, 0, 1, 1, 0, 0 ],  
  [ 1, 0, 0, 1, 0, 1, 0, 0, 1, 0 ], [ 1, 0, 0, 1, 0, 1, 0, 1, 0, 0 ],  
  [ 1, 0, 0, 1, 1, 0, 1, 0, 1, 1 ], [ 1, 0, 0, 1, 1, 0, 1, 1, 0, 1 ],  
  [ 1, 0, 0, 1, 1, 1, 0, 0, 1, 1 ], [ 1, 0, 0, 1, 1, 1, 0, 1, 0, 1 ] ]  
gap> p:=cc[1]*bp;  
[ [ 0, 1, 0, 0, 1, 1, 1, 1, 0, 0 ],  
  [ 1, 0, 1, 1, 0, 0, 0, 0, 1, 1 ],  
  [ 0, 0, 0, 0, 0, 1, 0, 1, 0, 0 ],  
  [ 0, 0, 0, 0, 1, 0, 1, 0, 0, 0 ],  
  [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 1 ],  
  [ 0, 1, 0, 0, 0, 0, 0, 1, 0, 0 ],  
  [ 0, 0, 1, 0, 0, 0, 0, 0, 1, 0 ],  
  [ 0, 1, 0, 0, 0, 0, 1, 0, 0, 0 ],  
  [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],  
  [ 1, 0, 0, 0, 0, 0, 0, 0, 1, 0 ] ]  
gap> Determinant(p);  
1  
gap> gg:=StablyPermutationFCheckGen(G,Nlist(l),Plist(l));  
[ [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],  
    [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],  
    [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ],  
    [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],  
    [ 0, 0, 0, 0, 0, 0, 0, 1, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 1 ], ],  
  [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],  
    [ 0, 0, 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
```

```

      [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 1, 0, -1, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ] ],
    [ [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ],
      [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],
      [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ] ] ],
    [ [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ] ],
      [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ], [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ] ],
      [ [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],
      [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ], [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ] ] ] ]
gap> List(gg[1],x->p^-1*x*p)=gg[2];
true
gap> StablyPermutationFCheckMat(G,Nlist(1),Plist(1),p);
true

```

### 5.7. Verification of $[M_G]^{fl} = 0$ : Method III.

Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank  $n$  as in Definition 1.24. In order to confirm that  $[M]^{fl} = 0$ , Method I and Method II in the previous two section gave how to find the explicit isomorphism

$$(10) \quad \left( \bigoplus_{i=1}^r a_i \mathbb{Z}[G/H_i] \right) \oplus b_1 F \simeq \left( \bigoplus_{i=1}^r a'_i \mathbb{Z}[G/H_i] \right) \oplus b'_1 F.$$

However, if the rank of the  $G$ -lattices in the both sides of (10) is large, the algorithms in Method I and Method II does not work. In this case, we should try to reduce the rank of the  $G$ -lattices in (10). The following algorithm (Algorithm F7) can search suitable base change of the permutation  $G$ -lattice  $P^\circ$  in a coflabby resolution  $0 \rightarrow \mathrm{Ker} f \rightarrow P^\circ \xrightarrow{f} M^\circ \rightarrow 0$  of  $M^\circ$  as in (3) in order to reduce the rank of  $\mathrm{Ker} f$ . Then we may get a “reduced” flabby resolution of  $M$ :  $0 \rightarrow M \rightarrow P \rightarrow (\mathrm{Ker} f)^\circ \rightarrow 0$ .

`SearchCoflabbyResolutionBase(G,b)` returns a list  $\mathcal{L} = \{m_1, \dots, m_s\}$  where the  $m_i$ ’s are all of the lists which satisfy that  $m_i = \mathcal{P}^\circ$ ,  $P^\circ = \mathbb{Z}[\mathcal{P}^\circ]$  and  $P^\circ$  satisfies (4) with  $\#\mathcal{R}' \leq b$ .

```

FlabbyResolutionFromBase(G,mi),
PossibilityOfStablyPermutationFFromBase(G,mi),
StablyPermutationFCheckFromBase(G,mi,L1,L2),
StablyPermutationFCheckPFromBase(G,mi,L1,L2),
StablyPermutationFCheckMatFromBase(G,mi,L1,L2,P)

```

return the same as

```

FlabbyResolution(G) (in Algorithm F1),
PossibilityOfStablyPermutationF(G) (in Algorithm F4),

```

StablyPermutationFCheck(G,L1,L2) (in Algorithm F5: Method I),  
 StablyPermutationFCheckP(G,L1,L2) (in Algorithm F6: Method II),  
 StablyPermutationFCheckMat(G,L1,L2,P) (in Algorithm F6: Method II)

respectively but with respect to  $m_i = \mathcal{P}^\circ$  instead of the original  $\mathcal{P}^\circ$  as in (4).

**Algorithm F7** (Base change of a flabby resolution of  $M_G$ : Method III).

```

CheckCoflabbyResolutionBaseH:= function(g,hh,mi)
  local z0;
  z0:=List(hh,Z0lattice);
  return ForAll([1..Length(hh)],i->LatticeBasis(List(OrbitsDomain(hh[i],mi),Sum))=z0[i]);
end;

CheckCoflabbyResolutionBase:= function(g,mi)
  local hh,z0;
  hh:=List(ConjugacyClassesSubgroups2(g),Representative);
  return CheckCoflabbyResolutionBaseH(g,hh,mi);
end;

SearchCoflabbyResolutionBaseH:= function(g,hh,b)
  local z0,orbs,imgs,i,j,mi,mis;
  mis=[];
  z0:=List(hh,Z0lattice);
  orbs:=Set(Union(z0),x->Orbit(g,x));
  imgs:=List(orbs,x->List(hh,y->LatticeBasis(List(OrbitsDomain(y,x),Sum))));
  if b=0 then
    for i in [1..Length(orbs)] do
      for j in Combinations([1..Length(orbs)],i) do
        if ForAll([1..Length(hh)],x->LatticeBasis(
          Union(List(j,y->imgs[y][x]))=z0[x]) then
          mi:=Union(List(j,x->orbs[x]));
          if mis=[] or Length(mi)<Length(mis) then
            mis:=mi;
          fi;
        fi;
      od;
      if mis<>[] then
        return mis;
      fi;
    od;
  else
    for i in [1..b] do
      for j in Combinations([1..Length(orbs)],i) do
        if ForAll([1..Length(hh)],x->LatticeBasis(
          Union(List(j,y->imgs[y][x]))=z0[x]) then
          mi:=Union(List(j,x->orbs[x]));
          Add(mis,mi);
        fi;
      od;
    od;
    return Set(mis);
  fi;
end;

```

```

SearchCoflabbyResolutionBase:= function(g,b)
  local hh;
  hh:=List(ConjugacyClassesSubgroups2(g),Representative);
  return SearchCoflabbyResolutionBaseH(g,hh,b);
end;

FlabbyResolutionFromBase:= function(g,mi)
  local tg,gg,d,th,ms,o,r,gg1,gg2,v1,mg,img;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  r:=Length(mi);
  o:=IdentityMat(r);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return rec(injection:=TransposedMat(mi),
              surjection:=NullMat(r,0),
              actionP:=TransposedMatrixGroup(Group(gg1,o))
            );
  else
    ms:=NullspaceIntMat(mi);
    v1:=NullspaceIntMat(TransposedMat(ms));
    mg:=Concatenation(v1,ms);
    img:=mg^-1;
    gg2:=List(gg1,x->mg*x*img);
    gg2:=List(gg2,x->x[[d+1..r]][[d+1..r]]);
    return rec(injection:=TransposedMat(mi),
              surjection:=TransposedMat(ms),
              actionP:=TransposedMatrixGroup(Group(gg1)),
              actionF:=TransposedMatrixGroup(Group(gg2))
            );
  fi;
end;

PossibilityOfStablyPermutationFFromBase:= function(g,mi)
  local tg,gg,d,th,ms,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
  tg:=TransposedMatrixGroup(g);
  gg:=GeneratorsOfGroup(tg);
  d:=Length(Identity(g));
  th:=List(ConjugacyClassesSubgroups2(tg),Representative);
  r:=Length(mi);
  gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
  if r=d then
    return [1,-1];
  else
    ms:=NullspaceIntMat(mi);
    v:=NullspaceIntMat(TransposedMat(ms));
    m1:=Concatenation(v,ms);
    m2:=m1^-1;
    gg2:=List(gg1,x->m1*x*m2);
    gg2:=List(gg2,x->x[[d+1..r]][[d+1..r]]);
    tg2:=Group(gg2);
    iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
    ker:=Kernel(iso);
  fi;
end;

```

```

    th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
    g2:=TransposedMatrixGroup(tg2);
    h2:=List(th2,TransposedMatrixGroup);
    return PossibilityOfStablyPermutationH(g2,h2);
fi;
end;

StablyPermutationFCheckFromBase:= function(g,mi,c1,c2)
    local tg,gg,d,th,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
    tg:=TransposedMatrixGroup(g);
    gg:=GeneratorsOfGroup(tg);
    d:=Length(Identity(g));
    th:=List(ConjugacyClassesSubgroups2(tg),Representative);
    r:=Length(mi);
    gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
    if r=d then
        return true;
    else
        ms:=NullspaceIntMat(mi);
        v:=NullspaceIntMat(TransposedMat(ms));
        m1:=Concatenation(v,ms);
        m2:=m1^-1;
        gg2:=List(gg1,x->m1*x*m2);
        gg2:=List(gg2,x->x[[d+1..r]][[d+1..r]]);
        tg2:=Group(gg2);
        iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
        ker:=Kernel(iso);
        th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
        g2:=TransposedMatrixGroup(tg2);
        h2:=List(th2,TransposedMatrixGroup);
        return StablyPermutationCheckH(g2,h2,c1,c2);
    fi;
end;

StablyPermutationFCheckPFromBase:= function(g,mi,c1,c2)
    local tg,gg,d,th,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
    tg:=TransposedMatrixGroup(g);
    gg:=GeneratorsOfGroup(tg);
    d:=Length(Identity(g));
    th:=List(ConjugacyClassesSubgroups2(tg),Representative);
    r:=Length(mi);
    gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
    if r=d then
        return true;
    else
        ms:=NullspaceIntMat(mi);
        v:=NullspaceIntMat(TransposedMat(ms));
        m1:=Concatenation(v,ms);
        m2:=m1^-1;
        gg2:=List(gg1,x->m1*x*m2);
        gg2:=List(gg2,x->x[[d+1..r]][[d+1..r]]);
        tg2:=Group(gg2);
        iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
        ker:=Kernel(iso);
        th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));

```

```

        g2:=TransposedMatrixGroup(tg2);
        h2:=List(th2,TransposedMatrixGroup);
        return StablyPermutationCheckHP(g2,h2,c1,c2);
    fi;
end;

StablyPermutationFCheckMatFromBase:= function(g,mi,c1,c2,p)
    local tg,gg,d,th,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
    tg:=TransposedMatrixGroup(g);
    gg:=GeneratorsOfGroup(tg);
    d:=Length(Identity(g));
    th:=List(ConjugacyClassesSubgroups2(tg),Representative);
    r:=Length(mi);
    gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
    if r=d then
        return true;
    else
        ms:=NullspaceIntMat(mi);
        v:=NullspaceIntMat(TransposedMat(ms));
        m1:=Concatenation(v,ms);
        m2:=m1^-1;
        gg2:=List(gg1,x->m1*x*m2);
        gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
        tg2:=Group(gg2);
        iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
        ker:=Kernel(iso);
        th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
        g2:=TransposedMatrixGroup(tg2);
        h2:=List(th2,TransposedMatrixGroup);
        return StablyPermutationCheckHMat(g2,h2,c1,c2,p);
    fi;
end;

StablyPermutationFCheckGenFromBase:= function(g,mi,c1,c2)
    local tg,gg,d,th,ms,o,h,r,gg1,gg2,g2,iso,ker,tg2,th2,h2,m1,m2,v;
    tg:=TransposedMatrixGroup(g);
    gg:=GeneratorsOfGroup(tg);
    d:=Length(Identity(g));
    th:=List(ConjugacyClassesSubgroups2(tg),Representative);
    r:=Length(mi);
    gg1:=List(gg,x->PermutationMat(Permutation(x,mi),r));
    if r=d then
        return true;
    else
        ms:=NullspaceIntMat(mi);
        v:=NullspaceIntMat(TransposedMat(ms));
        m1:=Concatenation(v,ms);
        m2:=m1^-1;
        gg2:=List(gg1,x->m1*x*m2);
        gg2:=List(gg2,x->x{[d+1..r]}{[d+1..r]});
        tg2:=Group(gg2);
        iso:=GroupHomomorphismByImages(tg,tg2,gg,gg2);
        ker:=Kernel(iso);
        th2:=List(Filtered(th,x->IsSubgroup(x,ker)),x->Image(iso,x));
        g2:=TransposedMatrixGroup(tg2);

```



```

h2:=List(th2,TransposedMatrixGroup);
return StablyPermutationCheckHGen(g2,h2,c1,c2);
fi;
end;

```

**Example 5.8** (Algorithm F7: Method III). Let  $M_G$  be the  $G$ -lattice where  $G \simeq C_2 \times S_4$  is the group of order 48 of the CARAT code (5,533,8). By using `FlabbyResolution(G).actionF` as in Algorithm F1, we obtain a flabby resolution  $0 \rightarrow M_G \rightarrow P \rightarrow F \rightarrow 0$  of  $M_G$ . However, the flabby  $G$ -lattice  $F$  is of rank 44.

By `FlabbyResolutionFromBase(G,mi).actionF`, we get a suitable flabby resolution of  $M_G$ :  $0 \rightarrow M_G \rightarrow P' \rightarrow F' \rightarrow 0$  where the flabby  $G$ -lattice  $F'$  is of rank 10.

By using `PossibilityOfStablyPermutationF(G)`, it may be possible that the isomorphism  $\mathbb{Z}[G/H_{20}] \oplus \mathbb{Z}[G/H_{22}] \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{29}] \oplus F'$  occurs where  $H_{20} \simeq C_2^3$ ,  $H_{22} \simeq D_4$  and  $H_{29} \simeq C_2 \times D_4$ .

In order to confirm the isomorphism, we will use Mersenne Twister (cf. [MN98]) to search an appropriate coefficients  $c_i$  to get a transformation matrix  $P = \sum_i c_i P_i$  which satisfies  $G_1 P = P G_2$  as in (10) (one can use the classical global random generator via `IsGlobalRandomSource` which is given in [Knu98, Algorithm A in 3.2.2 with lag 30]). We should choose suitable integers  $n_1, n_2, n_3$  in

```
rr:=List([1..n1],x->List([1..20],y->Random(rs,[n2..n3])))
```

to get the desired coefficients.

```

Read("caratnumber.gap");
Read("FlabbyResolution.gap");

```

```

gap> G:=CaratMatGroupZClass(5,533,8);; # G=C2xS4
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 44
44
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 29, 29, 15, 15, 15, 15 ]
gap> mi:=mis[3];
[ [ -1, -1, 0, -1, 0 ], [ -1, -1, 0, 0, -1 ], [ 0, -1, 0, -1, 0 ],
  [ 0, -1, 0, 0, -1 ], [ 0, -1, 1, -1, 0 ], [ 0, -1, 1, 0, -1 ],
  [ 0, 0, 0, -1, 0 ], [ 0, 0, 0, -1, 1 ], [ 0, 0, 0, 0, -1 ],
  [ 0, 0, 0, 1, -1 ], [ 0, 0, 1, -1, 0 ], [ 0, 0, 1, 0, -1 ],
  [ 0, 1, -1, 1, 1 ], [ 1, 0, 1, -1, 0 ], [ 1, 0, 1, 0, -1 ] ]
gap> FF:=FlabbyResolutionFromBase(G,mi).actionF;
<matrix group with 2 generators>
gap> Rank(FF.1); # FF is of rank 10 (=15-5)
10
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);
[ [ 1, 0, 0, 0, 0, -2, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 1, 0, 0,
    1, -1, 0, 0, 1, 1, -1, 1, -1, -1 ],
  [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, 0, -1, 0, 0, 0, 1, 0, 0,
    0, 0, 0, 2, 1, 1, 0, 0, -1, -1 ],
  [ 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 0, 0,
    1, -1, 0, 0, 0, 0, -1, 1, 0, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0,
    -1, -1, -1, 0, 1, 1, 1, 1, -1, -1 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0,
    0, 0, 0, 0, -1, 0, 0, 0, 1, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
  0, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, -1 ]
gap> Length(l);
34

```

```

gap> [1[20],1[22],1[33],1[29],1[34]];
[ 1, 1, 1, -1, -1 ]
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C2", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C2 x C2", "C2 x C2",
  "C2 x C2", "C4", "C4", "C2 x C2", "C2 x C2", "C6", "S3", "S3", "C2 x C2 x C2", "D8",
  "D8", "C2 x C2 x C2", "C4 x C2", "D8", "D8", "A4", "D12", "C2 x D8", "C2 x A4",
  "S4", "S4", "C2 x S4" ]
gap> [ss[20],ss[22],ss[33],ss[29]];
[ "C2 x C2 x C2", "D8", "C2 x S4", "C2 x D8" ]
gap> bp:=StablyPermutationFCheckPFFromBase(G,mi,Nlist(1),Plist(1));;
gap> Length(bp);
20
gap> Length(bp[1]); # rank of the both sides of (10) is 13
13
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..1000],x->List([1..20],y->Random(rs,[0..1])));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1); # MT found 3 solutions
[ [ 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1 ],
  [ 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1 ],
  [ 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1 ] ]
gap> p:=last[1]*bp;
[ [ 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1 ],
  [ 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1 ],
  [ 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1 ],
  [ 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1 ] ]
gap> Determinant(p);
1
gap> StablyPermutationFCheckMatFromBase(G,mi,Nlist(1),Plist(1),p);
true

gap> rs:=RandomSource(IsGlobalRandomSource); # alternatively
<RandomSource in IsGlobalRandomSource>
gap> rr:=List([1..1000],x->List([1..20],y->Random(rs,[0..1])));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1); # found 1 solution
[ [ 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1 ] ]

```

## 6. FLABBY AND COFLABBY $G$ -LATTICES

In this section, we will determine all the flabby and coflabby  $G$ -lattices  $M$  of rank  $M \leq 6$  by using the algorithms `IsFlabby` and `IsCoflabby` which are given in Section 5.

First we make a list of all  $\mathbb{Z}$ -class of  $GL(n, \mathbb{Z})$  and we filter off the groups  $G$  such that  $H^{-1}(G) \neq 0$  or  $H^1(G) \neq 0$  from the list. Next we use the derived subgroup  $D(G) = [G, G]$ , the center  $Z(G)$  and a 2-Sylow subgroup  $Sy_2(G)$  of  $G$  to filter the groups off. Finally we filter off the groups which are not flabby or not coflabby. Because `IsFlabby(G)` and `IsCoflabby(G)` are much slower than `Hminus1(G)` and `H1(G)`, we use `Hminus(G)` and `H1(G)` to the specified subgroups of  $G$  as above. The following algorithms are available from

<http://math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/> as KS.gap.

`AllFlabbyCoflabbyZClasses(n)` returns the list of all the GAP codes of  $G$  such that  $M_G$  is a flabby and coflabby  $G$ -lattice of rank  $n$  when  $2 \leq n \leq 4$ .

`AllFlabbyCoflabbyZClasses(n:Carat)` returns the same as `AllFlabbyCoflabbyZClasses(n)` but using the CARAT code instead of the GAP code when  $1 \leq n \leq 6$ .

`AllPermutationZClasses(n)` returns the list of all the GAP codes of  $G$  such that  $M_G$  is a permutation  $G$ -lattice when  $2 \leq n \leq 4$ .

`AllPermutationZClasses(n:Carat)` returns the same as `AllPermutation(n)` but using the CARAT code instead of the GAP code when  $1 \leq n \leq 6$ .

**Algorithm FC** (Flabby and coflabby  $G$ -lattices).

```

AllFlabbyCoflabbyZClasses:= function(n)
  local glnz,listg;
  glnz:=Concatenation(List([1..Length(cryst[n])],
    x->List([1..Length(cryst[n][x])],y->[n,x,y])));
  listg:=List(glnz,x->CaratMatGroupZClass(x[1],x[2],x[3]));
  listg:=Filtered(listg,
    x->Product(Hminus1(x))=1 and Product(H1(x))=1);
  listg:=Filtered(listg,
    x->ForAll([DerivedSubgroup(x),Centre(x),SylowSubgroup(x,2)],
      y->Product(Hminus1(y))=1 and Product(H1(y))=1));
  listg:=Filtered(listg,
    x->ForAll(List(ConjugacyClassesSubgroups2(x),Representative),
      y->Product(Hminus1(y))=1 and Product(H1(y))=1));
  if ValueOption("carat")=true or ValueOption("Carat")=true then
    return Set(listg,CaratZClass);
  else
    return Set(listg,CrystCatZClass);
  fi;
end;

AllPermutationZClasses:= function(n)
  local Sn,Snsub;
  Sn:=Group(List(GeneratorsOfGroup(SymmetricGroup(n)),
    x->PermutationMat(x,n)));
  Snsub:=List(ConjugacyClassesSubgroups2(Sn),Representative);
  if ValueOption("carat")=true or ValueOption("Carat")=true then
    return Set(Snsub,CaratZClass);
  else
    return Set(Snsub,CrystCatZClass);
  fi;
end;

```

**Example 6.1** (All flabby and coflabby  $G$ -lattices of rank  $n \leq 6$  which are not permutation). We may compute all the permutation  $G$ -lattices of rank  $n \leq 6$  and all the flabby and coflabby  $G$ -lattices by using Algorithm 6.

There exist 2 (resp. 4, 11, 19, 56) permutation  $G$ -lattices of rank 2 (resp. 3, 4, 5, 6). All the permutation  $G$ -lattices of rank  $2 \leq n \leq 5$  are given as follows:

$G$ : permutation	$\{1\}$	$C_2$
GAP code	(2, 1, 1, 1)	(2, 2, 1, 2)

$G$ : permutation	$\{1\}$	$C_2$	$C_3$	$S_3$		
GAP code	(3, 1, 1, 1)	(3, 2, 2, 2)	(3, 5, 1, 1)	(3, 5, 4, 1)		
$G$ : permutation	$\{1\}$	$C_2$	$C_2$	$C_2^2$	$C_2^2$	$C_3$
GAP code	(4, 1, 1, 1)	(4, 2, 1, 2)	(4, 3, 1, 3)	(4, 4, 1, 5)	(4, 5, 1, 1)	(4, 8, 1, 1)
$G$ : permutation	$S_3$	$C_4$	$D_4$	$A_4$	$S_4$	
GAP code	(4, 8, 3, 1)	(4, 12, 1, 1)	(4, 12, 3, 1)	(4, 24, 1, 1)	(4, 24, 3, 1)	
$G$ : permutation	$\{1\}$	$C_2$	$C_2$	$C_2^2$	$C_2^2$	$C_4$ $D_4$
CARAT code	(5, 1, 1)	(5, 4, 2)	(5, 7, 3)	(5, 9, 6)	(5, 19, 18)	(5, 58, 8) (5, 62, 8)
$G$ : permutation	$C_3$	$S_3$	$D_6$	$S_3$	$C_6$	$A_4$
CARAT code	(5, 181, 2)	(5, 186, 2)	(5, 192, 6)	(5, 218, 8)	(5, 220, 4)	(5, 502, 3)
$G$ : permutation	$C_4$	$D_5$	$C_5$	$S_5$	$F_{20}$	$A_5$
CARAT code	(5, 506, 6)	(5, 901, 3)	(5, 909, 2)	(5, 911, 3)	(5, 918, 3)	(5, 931, 3)

```
Read("caratnumber.gap");
```

```
Read("KS.gap");
```

```
gap> l2f:=AllFlabbyCoflabbyZClasses(2);
```

```
[ [ 2, 1, 1, 1 ], [ 2, 2, 1, 2 ] ]
```

```
gap> l2p:=AllPermutationZClasses(2);
```

```
[ [ 2, 1, 1, 1 ], [ 2, 2, 1, 2 ] ]
```

```
gap> l3f:=AllFlabbyCoflabbyZClasses(3);
```

```
[ [ 3, 1, 1, 1 ], [ 3, 2, 2, 2 ], [ 3, 5, 1, 1 ], [ 3, 5, 4, 1 ] ]
```

```
gap> l3p:=AllPermutationZClasses(3);
```

```
[ [ 3, 1, 1, 1 ], [ 3, 2, 2, 2 ], [ 3, 5, 1, 1 ], [ 3, 5, 4, 1 ] ]
```

```
gap> l4f:=AllFlabbyCoflabbyZClasses(4);
```

```
[ [ 4, 1, 1, 1 ], [ 4, 2, 1, 2 ], [ 4, 3, 1, 3 ], [ 4, 4, 1, 5 ], [ 4, 5, 1, 1 ],  
  [ 4, 8, 1, 1 ], [ 4, 8, 3, 1 ], [ 4, 12, 1, 1 ], [ 4, 12, 3, 1 ], [ 4, 14, 2, 2 ],  
  [ 4, 14, 3, 3 ], [ 4, 14, 3, 4 ], [ 4, 14, 8, 2 ], [ 4, 24, 1, 1 ], [ 4, 24, 3, 1 ] ]
```

```
gap> Length(l4f);
```

```
15
```

```
gap> l4p:=AllPermutationZClasses(4);
```

```
gap> Length(l4p);
```

```
11
```

```
gap> Difference(l4f,l4p);
```

```
[ [ 4, 14, 2, 2 ], [ 4, 14, 3, 3 ], [ 4, 14, 3, 4 ], [ 4, 14, 8, 2 ] ]
```

```
gap> l5f:=AllFlabbyCoflabbyZClasses(5:Carat);
```

```
[ [ 5, 1, 1 ], [ 5, 4, 2 ], [ 5, 7, 3 ], [ 5, 9, 6 ], [ 5, 19, 18 ], [ 5, 58, 8 ],  
  [ 5, 62, 8 ], [ 5, 181, 2 ], [ 5, 186, 2 ], [ 5, 192, 6 ], [ 5, 218, 4 ], [ 5, 218, 8 ],  
  [ 5, 220, 4 ], [ 5, 502, 3 ], [ 5, 506, 6 ], [ 5, 901, 3 ], [ 5, 909, 2 ],  
  [ 5, 911, 3 ], [ 5, 911, 4 ], [ 5, 918, 3 ], [ 5, 918, 4 ], [ 5, 931, 3 ], [ 5, 931, 4 ] ]
```

```
gap> Length(l5f);
```

```
23
```

```
gap> l5p:=AllPermutationZClasses(5:Carat);
```

```
gap> Length(l5p);
```

```
19
```

```
gap> Difference(l5f,l5p);
```

```
[ [ 5, 218, 4 ], [ 5, 911, 4 ], [ 5, 918, 4 ], [ 5, 931, 4 ] ]
```

```
gap> l6f:=AllFlabbyCoflabbyZClasses(6:Carat);
```

```
[ [ 6, 2, 2 ], [ 6, 4, 3 ], [ 6, 8, 6 ], [ 6, 11, 4 ], [ 6, 15, 12 ], [ 6, 159, 14 ],  
  [ 6, 161, 14 ], [ 6, 161, 28 ], [ 6, 197, 14 ], [ 6, 226, 14 ], [ 6, 226, 40 ],
```

```

[ 6, 231, 39 ], [ 6, 238, 27 ], [ 6, 246, 21 ], [ 6, 366, 27 ], [ 6, 891, 7 ],
[ 6, 894, 6 ], [ 6, 927, 9 ], [ 6, 984, 6 ], [ 6, 993, 16 ], [ 6, 1087, 20 ],
[ 6, 1090, 18 ], [ 6, 1142, 8 ], [ 6, 1199, 16 ], [ 6, 1968, 3 ], [ 6, 2007, 2 ],
[ 6, 2010, 3 ], [ 6, 2026, 6 ], [ 6, 2043, 4 ], [ 6, 2043, 8 ], [ 6, 2044, 4 ],
[ 6, 2051, 9 ], [ 6, 2068, 6 ], [ 6, 2069, 6 ], [ 6, 2069, 12 ], [ 6, 2070, 12 ],
[ 6, 2079, 14 ], [ 6, 2079, 28 ], [ 6, 2088, 18 ], [ 6, 2105, 12 ], [ 6, 2154, 26 ],
[ 6, 2156, 40 ], [ 6, 2156, 80 ], [ 6, 2188, 39 ], [ 6, 2263, 6 ], [ 6, 2278, 8 ],
[ 6, 2709, 1 ], [ 6, 2958, 3 ], [ 6, 2966, 2 ], [ 6, 2968, 4 ], [ 6, 2969, 4 ],
[ 6, 2969, 8 ], [ 6, 2977, 6 ], [ 6, 3046, 3 ], [ 6, 3053, 5 ], [ 6, 3066, 3 ],
[ 6, 3068, 7 ], [ 6, 3068, 8 ], [ 6, 3071, 7 ], [ 6, 3071, 8 ], [ 6, 3073, 7 ],
[ 6, 3073, 8 ], [ 6, 3073, 15 ], [ 6, 3073, 16 ], [ 6, 3076, 7 ], [ 6, 3076, 8 ],
[ 6, 3091, 11 ], [ 6, 3091, 12 ], [ 6, 3276, 9 ], [ 6, 3297, 9 ], [ 6, 3299, 9 ],
[ 6, 3302, 9 ], [ 6, 3575, 8 ], [ 6, 3662, 8 ], [ 6, 3663, 12 ], [ 6, 3749, 10 ],
[ 6, 4618, 18 ], [ 6, 4618, 19 ], [ 6, 4621, 18 ], [ 6, 4630, 52 ], [ 6, 4647, 101 ],
[ 6, 4722, 8 ], [ 6, 4733, 8 ], [ 6, 4743, 13 ], [ 6, 4750, 13 ], [ 6, 4762, 41 ],
[ 6, 4807, 41 ], [ 6, 4811, 41 ], [ 6, 4814, 82 ], [ 6, 4898, 3 ], [ 6, 4904, 3 ],
[ 6, 4915, 20 ], [ 6, 4919, 20 ], [ 6, 4929, 11 ], [ 6, 5210, 8 ], [ 6, 5210, 14 ],
[ 6, 5262, 11 ], [ 6, 5311, 16 ], [ 6, 5318, 8 ], [ 6, 5321, 6 ], [ 6, 5321, 14 ],
[ 6, 5421, 6 ], [ 6, 5424, 16 ], [ 6, 5475, 6 ], [ 6, 5477, 11 ], [ 6, 5487, 11 ] ]
gap> Length(l6f);
106
gap> l6p:=AllPermutationZClasses(6:Carat);
gap> Length(l6p);
56
gap> Difference(l6f,l6p);
[ [ 6, 159, 14 ], [ 6, 161, 14 ], [ 6, 161, 28 ], [ 6, 197, 14 ], [ 6, 226, 14 ],
[ 6, 226, 40 ], [ 6, 231, 39 ], [ 6, 238, 27 ], [ 6, 246, 21 ], [ 6, 366, 27 ],
[ 6, 1087, 20 ], [ 6, 1090, 18 ], [ 6, 1142, 8 ], [ 6, 2043, 4 ], [ 6, 2051, 9 ],
[ 6, 2068, 6 ], [ 6, 2069, 6 ], [ 6, 2069, 12 ], [ 6, 2070, 12 ], [ 6, 2079, 14 ],
[ 6, 2079, 28 ], [ 6, 2088, 18 ], [ 6, 2105, 12 ], [ 6, 2154, 26 ], [ 6, 2156, 40 ],
[ 6, 2156, 80 ], [ 6, 2188, 39 ], [ 6, 2968, 4 ], [ 6, 2969, 4 ], [ 6, 2969, 8 ],
[ 6, 2977, 6 ], [ 6, 3068, 7 ], [ 6, 3068, 8 ], [ 6, 3071, 7 ], [ 6, 3071, 8 ],
[ 6, 3073, 7 ], [ 6, 3073, 8 ], [ 6, 3073, 15 ], [ 6, 3073, 16 ], [ 6, 3076, 7 ],
[ 6, 3076, 8 ], [ 6, 3091, 11 ], [ 6, 3091, 12 ], [ 6, 5210, 14 ], [ 6, 5262, 11 ],
[ 6, 5321, 6 ], [ 6, 5421, 6 ], [ 6, 5475, 6 ], [ 6, 5477, 11 ], [ 6, 5487, 11 ] ]
gap> Length(last);
50

```

**Theorem 6.2.** *Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24.*

- (i) *When  $n \leq 3$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation.*
- (ii) *When  $n = 4$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation or the GAP code of  $G$  is one of  $(4, 14, 2, 2)$ ,  $(4, 14, 3, 3)$ ,  $(4, 14, 3, 4)$ ,  $(4, 14, 8, 2)$ .  
(There are 11 conjugacy classes of subgroups of  $S_4$  and hence 15 flabby and coflabby  $G$ -lattices of rank 4 in total.)*
- (iii) *When  $n = 5$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation or the CARAT code of  $G$  is one of  $(5, 218, 4)$ ,  $(5, 911, 4)$ ,  $(5, 918, 4)$ ,  $(5, 931, 4)$ .  
(There are 19 conjugacy classes of subgroups of  $S_5$  and hence 23 flabby and coflabby  $G$ -lattices of rank 5 in total.)*
- (iv) *When  $n = 6$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is permutation or the CARAT code of  $G$  is one of the 50 triples*

(6, 159, 14),	(6, 161, 14),	(6, 161, 28),	(6, 197, 14),	(6, 226, 14),
(6, 226, 40),	(6, 231, 39),	(6, 238, 27),	(6, 246, 21),	(6, 366, 27),
(6, 1087, 20),	(6, 1090, 18),	(6, 1142, 8),	(6, 2043, 4),	(6, 2051, 9),
(6, 2068, 6),	(6, 2069, 6),	(6, 2069, 12),	(6, 2070, 12),	(6, 2079, 14),
(6, 2079, 28),	(6, 2088, 18),	(6, 2105, 12),	(6, 2154, 26),	(6, 2156, 40),
(6, 2156, 80),	(6, 2188, 39),	(6, 2968, 4),	(6, 2969, 4),	(6, 2969, 8),
(6, 2977, 6),	(6, 3068, 7),	(6, 3068, 8),	(6, 3071, 7),	(6, 3071, 8),
(6, 3073, 7),	(6, 3073, 8),	(6, 3073, 15),	(6, 3073, 16),	(6, 3076, 7),
(6, 3076, 8),	(6, 3091, 11),	(6, 3091, 12),	(6, 5210, 14),	(6, 5262, 11),
(6, 5321, 6),	(6, 5421, 6),	(6, 5475, 6),	(6, 5477, 11),	(6, 5487, 11).

(There are 56 conjugacy classes of subgroups of  $S_6$  and hence 106 flabby and coflabby  $G$ -lattices of rank 6 in total.)

By Theorem 1.18, when any  $p$ -Sylow subgroup of  $G$  is cyclic for odd  $p$  and cyclic or dihedral (including Klein's four group) for  $p = 2$ ,  $G$ -lattice  $M$  is flabby and coflabby if and only if  $M$  is invertible. Moreover, for rank  $M \leq 6$ ,  $M$  is flabby and coflabby if and only if  $M$  is stably permutation.

**Theorem 6.3.** *Let  $G$  be a finite subgroup of  $\mathrm{GL}(n, \mathbb{Z})$  and  $M_G$  be the  $G$ -lattice as in Definition 1.24. When  $n \leq 6$ ,  $M_G$  is flabby and coflabby if and only if  $M_G$  is stably permutation. Indeed, flabby and coflabby  $G$ -lattices  $M_G$  which are not permutation as in Theorem 6.2 are stably permutation as in Table 8.*

Table 8: flabby and coflabby  $G$ -lattices  $M = M_G$  of rank  $\leq 6$  which are not permutation

GAP/CARAT code	$G$	$M$ is stably permutation, $H_i$ is the $i$ th conjugacy class of subgroups of $G$	
(4,14,2,2)	$C_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3]$	
(4,14,3,3)	$S_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3]$	
(4,14,3,4)	$S_3$	$M \oplus \mathbb{Z}[G/H_2] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}$	
(4,14,8,2)	$D_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_9]$	
(5,218,4)	$S_3$	$M \oplus \mathbb{Z}[G/H_2] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}^{\oplus 2}$	
(5,911,4)	$S_5$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{17}] \oplus \mathbb{Z}$	
(5,918,4)	$F_{20}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_3] \oplus \mathbb{Z}$	
(5,931,4)	$A_5$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_8] \oplus \mathbb{Z}$	
(6,159,14)	$C_{12}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_4]$	
(6,161,14)	$Q_{12}$	$M \oplus \mathbb{Z}[G/H_4] \oplus \mathbb{Z}[G/H_5] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3] \oplus \mathbb{Z}$	
(6,161,28)	$Q_{12}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_4]$	
(6,197,14)	$D_4 \times C_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{12}]$	
(6,226,14)	$C_3 \rtimes D_4$	$M \oplus \mathbb{Z}[G/H_{12}] \oplus \mathbb{Z}[G/H_{13}] \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}$	
(6,226,40)	$C_3 \rtimes D_4$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{12}]$	
(6,231,39)	$D_{12}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{12}]$	
(6,238,27)	$C_3 \rtimes D_4$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{11}] \oplus \mathbb{Z}[G/H_{12}]$	
(6,246,21)	$S_3 \times C_4$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{11}] \oplus \mathbb{Z}[G/H_{12}]$	
(6,366,27)	$D_4 \times S_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{43}] \oplus \mathbb{Z}[G/H_{46}]$	
(6,1087,20)	$S_3 \times C_3$	$M \oplus \mathbb{Z}[G/H_7] \simeq \mathbb{Z}[G/H_4] \oplus \mathbb{Z}[G/H_6]$	
(6,1090,18)	$C_3^2 \rtimes C_2$	$M \oplus \mathbb{Z}[G/H_9] \simeq \mathbb{Z}[G/H_5] \oplus \mathbb{Z}[G/H_{10}]$	
(6,1142,8)	$S_3 \times S_3$	$M \oplus \mathbb{Z}[G/H_{17}] \simeq \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{18}]$	
(6,2043,4)	$S_3$	$M \oplus \mathbb{Z}[G/H_2] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}^{\oplus 3}$	
(6,2051,9)	$D_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_9]^{\oplus 2}$	
(6,2068,6)	$C_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3]^{\oplus 2}$	
(6,2069,6)	$S_3$	$M \oplus \mathbb{Z}[G/H_2] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}[G/H_3] \oplus \mathbb{Z}$	
(6,2069,12)	$S_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3]^{\oplus 2}$	
(6,2070,12)	$C_6 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_8] \oplus \mathbb{Z}[G/H_9]$	
(6,2079,14)	$D_6$	$M \oplus \mathbb{Z}[G/H_6] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_9] \oplus \mathbb{Z}$	
(6,2079,28)	$D_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_7] \oplus \mathbb{Z}[G/H_9]$	
(6,2088,18)	$D_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_8] \oplus \mathbb{Z}[G/H_9]$	
(6,2105,12)	$D_6 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{24}] \oplus \mathbb{Z}[G/H_{29}] \oplus \mathbb{Z}[G/H_{31}]$	
(6,2154,26)	$C_6 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_5] \oplus \mathbb{Z}[G/H_6]$	
(6,2156,40)	$D_6$	$M \oplus \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_9] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_5] \oplus \mathbb{Z}$	
(6,2156,80)	$D_6$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_5] \oplus \mathbb{Z}[G/H_6]$	
(6,2188,39)	$D_6 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{19}] \oplus \mathbb{Z}[G/H_{24}]$	
(6,2968,4)	$C_{10}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3]$	
(6,2969,4)	$D_5$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3]$	
(6,2969,8)	$D_5$	$M \oplus \mathbb{Z}[G/H_2] \simeq \mathbb{Z}[G] \oplus \mathbb{Z}$	
(6,2977,6)	$D_{10}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_5] \oplus \mathbb{Z}[G/H_8]$	
(6,3068,7)	$A_5 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{20}] \oplus \mathbb{Z}[G/H_{21}]$	
(6,3068,8)	$A_5 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{20}] \oplus \mathbb{Z}[G/H_{21}]$	
(6,3071,7)	$S_5$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{17}] \oplus \mathbb{Z}[G/H_{18}]$	
(6,3071,8)	$S_5$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{17}] \oplus \mathbb{Z}[G/H_{18}]$	
(6,3073,7)	$F_{20}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_5]$	
(6,3073,8)	$F_{20}$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_5]$	
(6,3073,15)	$F_{20}$	$M \oplus \mathbb{Z}[G/H_3] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}$	
(6,3073,16)	$F_{20}$	$M \oplus \mathbb{Z}[G/H_3] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}$	
(6,3076,7)	$S_5 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{52}] \oplus \mathbb{Z}[G/H_{56}]$	
(6,3076,8)	$S_5 \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{52}] \oplus \mathbb{Z}[G/H_{56}]$	
(6,3091,11)	$F_{20} \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_9] \oplus \mathbb{Z}[G/H_{15}]$	
(6,3091,12)	$F_{20} \times C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_9] \oplus \mathbb{Z}[G/H_{15}]$	
(6,5210,14)	$A_4$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_4]$	
(6,5262,11)	$S_4 \times S_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{64}] \oplus \mathbb{Z}[G/H_{65}]$	
(6,5321,6)	$S_4$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_8] \oplus \mathbb{Z}[G/H_9]$	
(6,5421,6)	$A_4 \times C_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_9] \oplus \mathbb{Z}[G/H_{13}]$	
(6,5475,6)	$S_4 \times C_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{20}] \oplus \mathbb{Z}[G/H_{21}]$	
(6,5477,11)	$A_4 \times S_3$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{22}] \oplus \mathbb{Z}[G/H_{24}]$	
(6,5487,11)	$(A_4 \times A_3) \rtimes C_2$	$M \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{24}] \oplus \mathbb{Z}[G/H_{28}]$	

Note that in Table 8,  $H_i$  is the  $i$ -th conjugacy class of subgroups of  $G$  which is determined by the function `ConjugacyClassesSubgroups2` in GAP (see Section 4).

All cases can be done by using Method I in Section 5.5 and Method II in Section 5.6. We give the following 3 typical examples instead of the full proof of Theorem 6.3.

**Example 6.4 (Method I (1)).** We use `PossibilityOfStablyPermutationM` as in Algorithm F4 to get a possibility of the isomorphism. Then we use `StablyPermutationMCheck` as in Algorithm F5: Method I to get the actual isomorphism.

Let  $G \simeq C_6$  be the group of the GAP code  $(4, 14, 2, 2)$  which is generated by the matrix  $\sigma = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$ .

There exist 4 conjugacy classes of subgroups  $\{1\}$ ,  $H_2$ ,  $H_3$  and  $G$  of  $G$  which are isomorphic to cyclic groups of order 1, 2, 3 and 6. Corresponding  $G$ -lattices  $M_G$  are isomorphic to  $\mathbb{Z}[G]$ ,  $\mathbb{Z}[G/\langle\sigma^3\rangle]$ ,  $\mathbb{Z}[G/\langle\sigma^2\rangle]$  and  $\mathbb{Z}$  of rank 6, 3, 2 and 1 respectively.

Let  $G_1$  (resp.  $G_2$ ) be the matrix representation group of the action of  $G$  on  $\mathbb{Z} \oplus M_G$  (resp.  $\mathbb{Z}[G/\langle\sigma^3\rangle] \oplus \mathbb{Z}[G/\langle\sigma^2\rangle]$ ). `StablyPermutationMCheck(G, Nlist(1), Plist(1))` shows that  $G_1 P = P G_2$  with the isomorphism

$$\mathbb{Z} \oplus M_G \simeq \mathbb{Z}[G/\langle\sigma^3\rangle] \oplus \mathbb{Z}[G/\langle\sigma^2\rangle]$$

where

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Similar examples for the groups  $S_3$ ,  $S_3$  and  $D_6$  of the CARAT codes  $(4, 14, 3, 3)$ ,  $(4, 14, 3, 4)$  and  $(4, 14, 8, 2)$  are given below.

```
Read("FlabbyResolution.gap");
```

```
gap> G:=MatGroupZClass(4,14,2,2);; # G=C6
gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 0, 1, 1, -1, -1 ] ]
gap> l:=ll[1];
[ 0, 1, 1, -1, -1 ]
gap> List(ConjugacyClassesSubgroups2(G), x->StructureDescription(Representative(x)));
[ "1", "C2", "C3", "C6" ]
gap> StablyPermutationMCheck(G, Nlist(1), Plist(1));
[ [ 1, 1, 1, 1, 1 ],
  [ 0, 0, -1, 0, -1 ],
  [ 0, 0, -1, -1, 0 ],
  [ 1, 0, -1, 0, 0 ],
  [ 0, -1, 1, 0, 0 ] ]
```

```
gap> G:=MatGroupZClass(4,14,3,3);; # G=S3
gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 0, 1, 1, -1, -1 ] ]
gap> l:=ll[1];
[ 0, 1, 1, -1, -1 ]
gap> List(ConjugacyClassesSubgroups2(G), x->StructureDescription(Representative(x)));
[ "1", "C2", "C3", "S3" ]
gap> StablyPermutationMCheck(G, Nlist(1), Plist(1));
[ [ 1, 1, 1, 1, 1 ],
  [ 0, 0, -1, 0, -1 ],
  [ 0, 0, -1, -1, 0 ],
  [ 1, 0, -1, 0, 0 ],
  [ 0, -1, 1, 0, 0 ] ]
```

```
gap> G:=MatGroupZClass(4,14,3,4);; # G=S3
```



```

gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 1, -1, 0, 1, -1 ] ]
gap> l:=ll[1];
[ 1, -1, 0, 1, -1 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C3", "S3" ]
gap> StablyPermutationMCheck(G,Nlist(l),Plist(l));
[ [ 0, 0, -1, 0, 0, -1, -1 ],
  [ -1, 0, 0, 0, -1, 0, -1 ],
  [ 0, -1, 0, -1, 0, 0, -1 ],
  [ 0, 0, -1, -1, 0, -1, -1 ],
  [ 1, 1, 0, 0, 1, 0, 1 ],
  [ 1, 0, -1, -1, 1, 0, 0 ],
  [ 0, -1, 1, 0, -1, 1, 0 ] ]

gap> G:=MatGroupZClass(4,14,8,2);; # G=D6
gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 1, -1, -1, -1, -1, 0, 1, 1, -1, 0, 2 ],
  [ 0, 0, 0, 0, 0, 1, 0, 0, 1, -1, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 1, 0, 0, 1, -1, -1 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C2", "C3", "C2 x C2", "C6", "S3", "S3", "D12" ]
gap> StablyPermutationMCheck(G,Nlist(l),Plist(l));
[ [ 1, 1, 1, 1, 1 ],
  [ 0, 0, -1, 0, -1 ],
  [ 0, 0, -1, -1, 0 ],
  [ 1, 0, -1, 0, 0 ],
  [ 0, -1, 1, 0, 0 ] ]

```

**Example 6.5 (Method I (2)).** Before applying `StablePermutationMCheck` in Method I, in some cases, we have to add some more  $G$ -lattices to make both hand side of  $G$ -lattices isomorphic. We will show that for the group  $G \simeq S_5$  of the CARAT code (5,911,4),  $M_G \not\simeq \mathbb{Z}[S_5/S_4]$  but  $M_G \oplus \mathbb{Z} \simeq \mathbb{Z}[S_5/S_4] \oplus \mathbb{Z}$ .

```

gap> Read("caratnumber.gap");
gap> Read("FlabbyResolution.gap");

gap> G:=CaratMatGroupZClass(5,911,4);; # G=S5
gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 1, 0, 0, -1, 0, 0, -4, 0, -2, 1, 2, 0, -1, -1, 0, 4, 0, 1, -4, 4 ],
  [ 0, 1, 0, 0, 0, -1, -1, 0, -1, 0, 0, 0, 0, 0, 1, 1, 0, 0, -1, 1 ],
  [ 0, 0, 1, 0, 0, 0, -2, 0, -1, 0, 1, 0, -1, -1, 0, 2, 0, 1, -2, 2 ],
  [ 0, 0, 0, 0, 1, 2, -2, 0, -2, 1, 2, -2, -1, -2, -2, 2, 0, 1, -2, 4 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C4", "C5", "S3", "S3", "C6", "D8",
  "D10", "A4", "D12", "C5 : C4", "S4", "A5", "S5" ]
gap> StablyPermutationMCheck(G,Nlist(l),Plist(l));
fail
gap> l2:=IdentityMat(Length(l))[Length(l)-1];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ]
gap> StablyPermutationMCheck(G,Nlist(l)+l2,Plist(l)+l2);
[ [ 2, 2, 2, 2, 2, 3 ],

```

```
[ 0, -1, 0, -1, -1, -1 ],
[ 0, 1, -1, 0, 0, 0 ],
[ 1, 1, 0, 0, 1, 1 ],
[ 1, 0, 1, 1, 0, 1 ],
[ -1, -2, -1, -1, -1, -2 ] ]
```

**Example 6.6 (Method II).** When `StablyPermutationMCheck` does not return any result in an appropriate time, we may use the command `StablyPermutatinoCheakP`.

```
gap> Read("caratnumber.gap");
gap> Read("FlabbyResolution.gap");

gap> G:=CaratMatGroupZClass(6,161,14);; # G=Q12
gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 0, 1, 1, -1, -1, 1, -1 ] ]
gap> l:=ll[1];
[ 0, 1, 1, -1, -1, 1, -1 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C3", "C4", "C6", "C3 : C4" ]
gap> gg:=StablyPermutationMCheckGen(G,Nlist(l),Plist(l));
[ [ [ [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, -1 ],
      [ 0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1 ] ],
  [ [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
    [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
    [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
    [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 0, -2, -1, 1, 1 ],
    [ 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0 ],
    [ 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1 ] ] ],
  [ [ [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ],
      [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ] ],
    [ [ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
      [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
```

```

[ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ] ] ] ]
gap> bp:=StablyPermutationMCheckP(G,Nlist(1),Plist(1));;
gap> Length(bp);
19
gap> Length(bp[1]); # rank of the both sides is 11
11
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..1000],x->List([1..19],y->Random(rs,[0,1])));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1);
[ [ 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1 ],
  [ 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1 ] ]
gap> p:=last[1]*bp;
[ [ 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1 ],
  [ 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1 ],
  [ 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1 ],
  [ 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0 ],
  [ 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 0 ],
  [ 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0 ],
  [ -1, 0, -1, 0, -1, 0, 0, 0, 0, -1, -1, 0 ],
  [ -1, 1, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0 ],
  [ 1, 0, 1, 1, 2, 1, 0, 1, 1, 0, 2, 0 ],
  [ 1, 2, 1, 1, 0, 1, 1, 0, 0, 1, 2, 0 ],
  [ -1, 0, -1, 1, 0, 1, -1, 1, 1, -1, 0, 0 ] ]
gap> Determinant(p);
1
gap> List(gg[1],x->p^-1*x*p)=gg[2];
true
gap> StablyPermutationMCheckMat(G,Nlist(1),Plist(1),p);
true

```

As an application of Example 6.4, we see that Krull-Schmidt theorem fails for permutation  $D_6$ -lattices by constructing explicit isomorphism. Let  $M_G$  be the  $G$ -lattice where  $G \simeq D_6 \leq \mathrm{GL}(4, \mathbb{Z})$  is the group of the GAP code (4, 14, 8, 2). Example 6.4 confirms that the isomorphism

$$(11) \quad M_G \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_9]$$

holds. We also see by Example 6.4 that it is possible that

$$(12) \quad 2M_G \oplus \mathbb{Z}[G] \oplus \mathbb{Z}[G/H_7] \oplus \mathbb{Z}[G/H_8] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_4] \oplus \mathbb{Z}[G/H_5] \oplus \mathbb{Z}[G/H_9].$$

If the isomorphism (12) holds, then it follows from (11) and (12) that Krull-Schmidt uniqueness fails for the permutation  $D_6$ -lattices (the rank of the both sides is  $12 + 2 \times 3 + 2 + 2 + 2 = 6 + 6 + 6 + 4 + 2 \times 1 = 24$ ):

$$\mathbb{Z}[G] \oplus 2\mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_7] \oplus \mathbb{Z}[G/H_8] \oplus \mathbb{Z}[G/H_9] \simeq \mathbb{Z}[G/H_2] \oplus \mathbb{Z}[G/H_3] \oplus \mathbb{Z}[G/H_4] \oplus \mathbb{Z}[G/H_5] \oplus 2\mathbb{Z}.$$

We may check that this isomorphism actually holds (see Example 6.8 below), namely we have:

**Proposition 6.7** (Krull-Schmidt uniqueness fails for permutation  $D_6$ -lattices). *Let  $D_6$  be the dihedral group of order 12 and  $\{1\}, C_2^{(1)}, C_2^{(2)}, C_2^{(3)}, C_3, C_2^2, C_6, S_3^{(1)}, S_3^{(2)}$  and  $D_6$  be the conjugacy classes of subgroups of  $D_6$ .*

Then the following isomorphism holds:

$$\begin{aligned} & \mathbb{Z}[D_6] \oplus 2\mathbb{Z}[D_6/C_2^2] \oplus \mathbb{Z}[D_6/C_6] \oplus \mathbb{Z}[D_6/S_3^{(1)}] \oplus \mathbb{Z}[D_6/S_3^{(2)}] \\ & \simeq \mathbb{Z}[D_6/C_2^{(1)}] \oplus \mathbb{Z}[D_6/C_2^{(2)}] \oplus \mathbb{Z}[D_6/C_2^{(3)}] \oplus \mathbb{Z}[D_6/C_3] \oplus 2\mathbb{Z}. \end{aligned}$$

**Example 6.8** (Krull-Schmidt uniqueness fails for permutation  $D_6$ -lattices).

```
Read("FlabbyResolution.gap");
```

```
gap> G:=MatGroupZClass(4,14,8,2);; # G=D6
gap> ll:=PossibilityOfStablyPermutationM(G);
[ [ 1, -1, -1, -1, -1, 0, 1, 1, -1, 0, 2 ],
  [ 0, 0, 0, 0, 0, 1, 0, 0, 1, -1, -1 ] ]
gap> l:=ll[1]+2*ll[2];
[ 1, -1, -1, -1, -1, 2, 1, 1, 1, -2, 0 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C2", "C3", "C2 x C2", "C6", "S3", "S3", "D12" ]
gap> bp:=StablyPermutationMCheckP(G,Nlist(l),Plist(l));;
gap> Length(bp);
68
gap> Length(bp[1]); # rank of the both sides is 24
24
```

# after some efforts we may get

```
gap> n:=[ 1, -1, -1, -1, -1, -1, -1, -1, 0, -1, 0, 1, 0, 1, 1, 0, 0, 0, -1, 1,
> 1, -1, 1, 0, -1, 1, -1, 0, -1, 0, 1, 1, 0, 1, 0, 0, -1, 1, 1, -1,
> -1, 0, -1, 0, -1, 1, 0, -1, -1, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0,
> 1, 1, 1, 1, 0, -1, -1, 0 ];;
gap> p:=n*bp;
[ [ 1, -1, -1, -1, -1, -1, 1, -1, -1, -1, -1, -1, -1, 0, -1, 0, 1, 0, 1, 1, 1, 0, 0 ],
  [ -1, 1, -1, -1, -1, -1, -1, 1, -1, -1, -1, -1, -1, 0, 0, -1, 1, 1, 0, 1, 1, 0, 0 ],
  [ -1, -1, 1, -1, -1, -1, -1, -1, 1, -1, -1, -1, 0, -1, -1, 1, 0, -1, 1, 0, 1, 1, 0, 0 ],
  [ -1, -1, -1, 1, -1, -1, -1, -1, -1, 1, -1, -1, 0, -1, -1, 1, -1, 0, 0, 1, 1, 1, 0, 0 ],
  [ -1, -1, -1, -1, 1, -1, -1, -1, -1, -1, 1, -1, -1, 0, -1, 0, 1, -1, 0, 1, 1, 1, 0, 0 ],
  [ -1, -1, -1, -1, -1, 1, -1, -1, -1, -1, -1, 1, -1, 0, -1, -1, 1, 0, 1, 0, 1, 1, 0, 0 ],
  [ 0, 0, -1, 1, 1, -1, -1, 1, 1, -1, 0, 0, 1, 0, 0, -1, 1, 1, -1, -1, 0, -1, 0, 0 ],
  [ 0, 0, 1, -1, -1, 1, 1, -1, 0, 0, 1, -1, 0, 1, 0, 1, -1, 1, -1, -1, -1, 0, 0, 0 ],
  [ -1, 1, 0, 0, -1, 1, 1, -1, -1, 1, 0, 0, 0, 0, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0 ],
  [ 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, -1, 1, 0, 1, 0, 1, -1, 1, -1, -1, 0, -1, 0, 0 ],
  [ 1, -1, -1, 1, 0, 0, -1, 1, 0, 0, -1, 1, 0, 0, 1, 1, 1, -1, -1, -1, 0, -1, 0, 0 ],
  [ -1, 1, 1, -1, 0, 0, 0, 0, -1, 1, 1, -1, 1, 0, 0, -1, 1, 1, -1, -1, -1, 0, 0, 0 ],
  [ 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, -1, 1, 1, 1, -1, -1, -1, -1, 0, 0, -1, 0 ],
  [ 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, -1, 1, -1, 1, -1, -1, -1, 0, 0, -1, 0 ],
  [ 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, -1, -1, -1, 1, -1, -1, 0, 0, -1, 0 ],
  [ 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, -1, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0, -1 ],
  [ 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, -1, 1, -1, 1, -1, -1, -1, 0, 0, 0, -1 ],
  [ 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, -1, -1, -1, 1, -1, -1, 0, 0, 0, -1 ],
  [ -1, 1, 1, -1, -1, 1, 0, -1, -1, 0, 0, -1, -1, -1, -1, 1, 1, 1, 1, 0, 0, 1, 0, 1 ],
  [ 1, -1, -1, 1, 1, -1, -1, 0, 0, -1, -1, 0, -1, -1, -1, 1, 1, 1, 0, 1, 1, 0, 0, 1 ],
  [ 0, -1, -1, 0, 0, -1, -1, 1, 1, -1, -1, 1, -1, -1, -1, 1, 1, 1, 1, 0, 1, 0, 1, 0 ],
  [ -1, 0, 0, -1, -1, 0, 1, -1, -1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 0, 1, 0, 1, 1, 0 ],
  [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1 ],
  [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, -1, -1, -1, -1, 0, 0 ] ]
```

```
gap> Determinant(p);
```

```
1
```

```
gap> StablyPermutationMCheckMat(G,Nlist(1),Plist(1),p);
true
```

## 7. NORM ONE TORI

Let  $K/k$  be a separable field extension of degree  $n$  and  $L/k$  be the Galois closure of  $K/k$ . Let  $G = \text{Gal}(L/k)$  and  $H = \text{Gal}(L/K)$ . The Galois group  $G$  may be regarded as a transitive subgroup of the symmetric group  $S_n$  of degree  $n$ . Let  $R_{K/k}^{(1)}(\mathbb{G}_m)$  be the norm one torus of  $K/k$ , i.e. the kernel of the norm map  $R_{K/k}(\mathbb{G}_m) \rightarrow \mathbb{G}_m$ . The norm one torus  $R_{K/k}^{(1)}(\mathbb{G}_m)$  has the Chevalley module  $J_{G/H}$  as its character module and the field  $L(J_{G/H})^G$  as its function field (see Section 1). The following algorithm is available from <http://math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/> as `FlabbyResolution.gap`.

`Norm1TorusJ(d,m)` returns the Chevalley module  $J_{G/H}$  for the  $m$ -th transitive subgroup  $G = dTm \leq S_d$  of degree  $d$  where  $H$  is the stabilizer of one of the letters in  $G$ .

**Algorithm N1T** (Construction of Chevalley module  $J_{G/H}$  for the transitive subgroups  $G = dTm \leq S_d$ ).

```
Norm1TorusJ:= function(d,m)
  local I,M1,M2,M,f,Sd,T;
  I:=IdentityMat(d-1);
  Sd:=SymmetricGroup(d);
  T:=TransitiveGroup(d,m);
  M1:=Concatenation(List([2..d-1],x->I[x]),[-List([1..d-1],One)]);
  if d=2 then
    M:=[M1];
  else
    M2:=Concatenation([I[2],I[1]],List([3..d-1],x->I[x]));
    M:=[M1,M2];
  fi;
  f:=GroupHomomorphismByImages(Sd,Group(M),GeneratorsOfGroup(Sd),M);
  return Image(f,T);
end;
```

**Example 7.1** ( $[J_{G/H}]^{fl} = 0$  for  $G = 5T4 \simeq A_5$ ). By using Method II as in Algorithm F6, we may verify that  $[J_{G/H}]^{fl} = 0$  for  $G = 5T4 \simeq A_5$  and  $H = A_4$ .

```
gap> Read("crystcat.gap");
gap> Read("FlabbyResolution.gap");

gap> J54:=Norm1TorusJ(5,4);
<matrix group with 2 generators>
gap> StructureDescription(J54);
"A5"
gap> CrystCatZClass(J54);
[ 4, 31, 3, 2 ]
gap> IsInvertibleF(J54);
true
gap> Rank(FlabbyResolution(J54).actionF.1); # F is of rank 16
16

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(J54),5);;
gap> Set(List(mis,Length))-4; # Method III could not apply
[ 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81 ]
```

```

gap> ll:=PossibilityOfStablyPermutationF(J54);
[ [ 1, -2, -1, 0, 0, 1, 1, 1, -1, 0 ], [ 0, 0, 0, 0, 1, 1, -1, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 1, 1, -1, 0, 0, -1 ]
gap> bp:=StablyPermutationFCheckP(J54,Nlist(l),Plist(l));;
gap> Length(bp);
11
gap> Length(bp[1]); # rank of the both sides of (10) is 22
22
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..10000],x->List([1..11],y->Random(rs,[-1..2]))));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1);
[ [ 2, 0, 0, -1, -1, 0, -1, 1, 0, 1, 1 ] ]
gap> p:=last[1]*bp;
[ [ 2, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, -1, -1 ],
  [ 0, 2, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, -1, 0, 0, 0, -1, -1, 0, -1, 0 ],
  [ 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, -1, 0, -1, 0, -1, 0, -1, -1 ],
  [ 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, -1, -1, 0, -1, 0, -1, -1, 0 ],
  [ 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, -1, -1, 0, -1 ],
  [ 0, 0, 0, 0, 0, 0, 2, 0, 0, 2, 0, 0, 0, 0, -1, -1, 0, -1, 0, 0, -1, 0, 0 ],
  [ -1, 0, -1, 1, 1, 0, -1, 1, -1, -1, 0, -1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1 ],
  [ 1, -1, 1, 1, -1, -1, -1, -1, 0, -1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1 ],
  [ -1, -1, 0, -1, 0, 1, 1, -1, -1, 0, -1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1 ],
  [ 1, 1, -1, -1, 0, 1, -1, 0, 0, -1, -1, -1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0 ],
  [ -1, 0, 0, -1, -1, -1, 1, 1, -1, 0, -1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0 ],
  [ 0, -1, -1, 0, -1, -1, 0, -1, 1, 1, 1, -1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0 ],
  [ -1, -1, 0, 0, 0, 1, -1, -1, -1, -1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1 ],
  [ -1, 1, 1, -1, -1, -1, 0, 0, -1, 1, -1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0 ],
  [ -1, 0, -1, 1, -1, -1, 1, 1, -1, 0, 0, -1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0 ],
  [ 0, -1, 0, 0, -1, -1, -1, -1, 1, -1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1 ],
  [ 0, -1, -1, -1, 1, 0, 0, -1, 1, 1, -1, -1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0 ],
  [ 0, 0, 0, -1, -1, -1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0 ],
  [ -1, 1, -1, 0, -1, -1, 0, 0, -1, 1, 1, -1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0 ],
  [ 1, -1, -1, 1, -1, -1, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0 ],
  [ 0, 0, -1, -1, 1, 0, -1, 1, 1, -1, -1, -1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0 ],
  [ 1, -1, -1, -1, 0, 1, 1, -1, 0, 0, -1, -1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0 ] ]
gap> Determinant(p);
-1
gap> StablyPermutationFCheckMat(J54,Nlist(l),Plist(l),p);
true

```

**Example 7.2** ( $[J_{G/H}]^{fl} = 0$  for  $G = 6T3 \simeq D_6$ ). By using Method II as in Algorithm F6, we may verify that  $[J_{G/H}]^{fl} = 0$  for  $G = 6T3 \simeq D_6$  and  $H = C_2$ .

```

gap> Read("caratnumber.gap");
gap> Read("FlabbyResolution.gap");

gap> J63:=Norm1TorusJ(6,3);
<matrix group with 2 generators>
gap> StructureDescription(J63);
"D12"
gap> CaratZClass(J63);
[ 5, 391, 4 ]
gap> IsInvertibleF(J63);

```

```

true
gap> Rank(FlabbyResolution(J63).actionF.1); # F is of rank 13
13
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(J63),3);; # Method III
gap> List(mis,Length);
[ 24, 20, 24, 18, 24, 20, 24, 20, 18, 14, 21, 17, 21, 24, 20, 24, 24, 20,
  24, 18, 30, 26, 24, 20, 18, 20, 14, 24, 20, 18 ]
gap> mi:=mis[Length(mis)-3]; # (new) F is of rank 9 (=14-5)
[ [ -1, 0, 0, 0, 0 ], [ -1, 1, -1, 1, -1 ], [ -1, 1, 0, 0, 0 ], [ 0, -1, 1, 0, 0 ],
  [ 0, 0, -1, 1, 0 ], [ 0, 0, 0, -1, 1 ], [ 0, 0, 0, 0, -1 ], [ 0, 0, 0, 0, 1 ],
  [ 0, 0, 0, 1, -1 ], [ 0, 0, 1, -1, 0 ], [ 0, 1, -1, 0, 0 ], [ 1, -1, 0, 0, 0 ],
  [ 1, -1, 1, -1, 1 ], [ 1, 0, 0, 0, 0 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(J63,mi);
[ [ 1, -1, 0, -1, 0, 2, 0, 1, 1, -1, -1 ], [ 0, 0, 1, 0, 1, 0, -1, 0, 0, 1, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 1, 0, 1, 0, -1, 0, 0, 1, -1 ]
gap> bp:=StablyPermutationFCheckPFromBase(J63,mi,Nlist(l),Plist(l));;
gap> Length(bp);
17
gap> Length(bp[1]); # rank of the both sides of (10) is 11
11
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..5000],x->List([1..17],y->Random(rs,[-1..1])));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1);
[ [ 0, 1, -1, 0, -1, 1, -1, 0, 1, 1, 0, -1, 1, -1, 0, 0, 1 ] ]
gap> p:=last[1]*bp;
[ [ 0, 1, 1, 0, 1, 0, -1, 0, 0, -1, -1 ],
  [ 1, 0, 0, 1, 0, 1, 0, -1, -1, 0, -1 ],
  [ 1, -1, 0, 1, -1, 1, 1, 0, -1, 1, -1 ],
  [ 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 1 ],
  [ 0, 1, 1, -1, 1, -1, -1, 1, 1, 0, -1 ],
  [ 1, 0, -1, 1, -1, 1, 1, -1, 0, 1, -1 ],
  [ -1, 1, 1, -1, 1, 0, -1, 1, 1, 0, -1 ],
  [ 1, -1, -1, 1, 0, 1, 1, -1, 0, 1, -1 ],
  [ -1, 1, 1, 0, 1, -1, -1, 1, 1, 0, -1 ],
  [ -1, 1, 1, 0, 1, -1, 0, 1, 1, -1, -1 ],
  [ 1, -1, -1, 1, 0, 1, 1, 0, -1, 1, -1 ] ]
gap> Determinant(p);
1
gap> StablyPermutationFCheckMatFromBase(J63,mi,Nlist(l),Plist(l),p);
true

```

**Example 7.3** (The flabby class  $[J_{Q_8}]^{fl}$  is not invertible but flabby and coflabby). Let  $J_{Q_8}$  be the Chevalley module of the quaternion group  $Q_8$  of order 8. The rank of  $J_{Q_8}$  is 7. Let  $8T5$  be the 5th transitive subgroup of  $S_8$ . Then  $8T5 \simeq Q_8$ . By Theorems 1.3 (i), 1.16 (ii) and 1.17, the flabby class  $[J_{Q_8}]^{fl}$  of  $J_{Q_8}$  is not invertible but flabby and coflabby. Using `FlabbyResolution` as in Algorithm F1, we may obtain the flabby class  $[J_{Q_8}]^{fl} = [F]$  of  $J_{Q_8}$  where  $F$  is of rank 9 which satisfies  $0 \rightarrow M_{Q_8} \rightarrow P \rightarrow F \rightarrow 0$ .

```

gap> Read("FlabbyResolution.gap");

gap> J85:=Norm1TorusJ(8,5);
<matrix group with 2 generators>
gap> StructureDescription(J85);
"Q8"

```

```

gap> F:=FlabbyResolution(J85).actionF;
<matrix group with 2 generators>
gap> Rank(F.1); # F is of rank 9
9

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(J85),5);;
gap> Set(List(mis,Length))-7; # Method III could not apply
[ 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33 ]

gap> IsFlabby(F);
true
gap> IsCoflabby(F);
true
gap> IsInvertibleF(J85);
false

```

**Example 7.4** (Norm one tori of dimensions 7 and 11). Note that by Theorems 1.3, 1.5, 1.6 and 1.7, and Lemma 2.17, we may obtain a birational classification of the norm one tori of dimensions 7 and 11 as follows:

- (i)  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is stably  $k$ -rational for  $7T1 \simeq C_7$  and  $7T2 \simeq D_7$ ;
- (ii)  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is not stably but retract  $k$ -rational for  $7T3 \simeq F_{21}$ ,  $7T4 \simeq F_{42}$ ,  $7T5 \simeq \text{PSL}(2, 7)$ ,  $7T6 \simeq A_7$  and  $7T7 \simeq S_7$ ;
- (iii)  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is stably  $k$ -rational for  $11T1 \simeq C_{11}$  and  $11T2 \simeq D_{11}$ ;
- (iv)  $R_{K/k}^{(1)}(\mathbb{G}_m)$  is not stably but retract  $k$ -rational for  $11T3 \simeq F_{55}$ ,  $11T4 \simeq F_{110}$ ,  $11T5 \simeq \text{PSL}(2, 11)$ ,  $11T6 \simeq M_{11}$ ,  $11T7 \simeq A_{11}$  and  $11T8 \simeq S_{11}$  where  $M_{11}$  is the Mathieu group of degree 11.

## 8. TATE COHOMOLOGY: GAP COMPUTATIONS

In this section, we provide some algorithms of GAP for computing the Tate cohomology  $\hat{H}^n(G, M_G)$  by using the GAP package HAP ([HAP]). We will use this for showing the main theorem (Theorem 1.25) in Section 9. The following algorithms are available from <http://math.h.kyoto-u.ac.jp/~yamasaki/Algorithm/asHn.gap>.

`TateCohomology(G,n)` returns the Tate cohomology group  $\hat{H}^n(G, M_G)$  for  $n \in \mathbb{Z}$ .

**Algorithm TC** (Tate cohomology).

```

LoadPackage("HAP");

CochainComplexMatrixGroup:= function(M,n)
  local IA,G,A,R,TR;
  IA:=RegularActionHomomorphism(TransposedMatrixGroup(M));
  G:=Image(IA);
  A:=InverseGeneralMapping(IA);
  R:=ResolutionFiniteGroup(G,n);
  TR:=HomToIntegralModule(R,A);
  return TR;
end;

ChainComplexMatrixGroup:= function(M,n)
  local IA,G,A,R,TR;
  IA:=RegularActionHomomorphism(TransposedMatrixGroup(M));
  G:=Image(IA);
  A:=InverseGeneralMapping(IA);

```



```

R:=ResolutionFiniteGroup(G,n);
TR:=TensorWithIntegralModule(R,A);
return TR;
end;

CochainComplexRightCosets:= function(g,h,n)
  local gg,hg,og,gp,A,R,TR;
  gg:=GeneratorsOfGroup(g);
  hg:=SortedList(RightCosets(g,h));
  og:=Length(hg);
  gp:=List(gg,x->PermutationMat(Permutation(x,hg,OnRight),og));
  A:=GroupHomomorphismByImages(g,Group(gp),gg,gp);
  R:=ResolutionFiniteGroup(g,n);
  TR:=HomToIntegralModule(R,A);
  return TR;
end;

ChainComplexRightCosets:= function(g,h,n)
  local gg,hg,og,gp,A,R,TR;
  gg:=GeneratorsOfGroup(g);
  hg:=SortedList(RightCosets(g,h));
  og:=Length(hg);
  gp:=List(gg,x->PermutationMat(Permutation(x,hg,OnRight),og));
  A:=GroupHomomorphismByImages(g,Group(gp),gg,gp);
  R:=ResolutionFiniteGroup(g,n);
  TR:=TensorWithIntegralModule(R,A);
  return TR;
end;

TateCohomology:= function(g,n)
  local m,s,r,TR;
  if n=0 then
    m:=Sum(g);
    s:=SmithNormalFormIntegerMat(m);
    r:=Rank(s);
    return List([1..r],x->s[x][x]);
  elif n>0 then
    TR:=CochainComplexMatrixGroup(g,n+1);
    return Cohomology(TR,n);
  else
    TR:=ChainComplexMatrixGroup(g,-n);
    return Homology(TR,-n-1);
  fi;
end;

```

**Example 8.1** (Tate Cohomology  $\widehat{H}^n(G, M_G)$  for the group  $C_2^3$  of the GAP code (3, 3, 3, 3)). Let  $G \simeq C_2^3$  be the group of the GAP code (3, 3, 3, 3). We may obtain the Tate cohomologies  $\widehat{H}^n(G, M_G)$  for  $-7 \leq n \leq 7$  as follows:

$n$	-7	-6	-5	-4	-3	-2	-1	0
$\widehat{H}^n(C_2^3, M_{C_2^3})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 13}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 9}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 7}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$n$	1	2	3	4	5	6		7
$\widehat{H}^n(C_2^3, M_{C_2^3})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 12}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 15}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 20}$	

```

gap> Read("Hn.gap");

gap> G:=MatGroupZClass(3,3,3,3);;
gap> List([-7..7],i->TateCohomology(G,i));
[ [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ], [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ],
  [ 2, 2, 2, 2, 2, 2, 2 ], [ 2, 2, 2, 2 ], [ 2, 2, 2 ], [ 2 ], [ 2 ], [  ],
  [ 2, 2 ], [ 2, 2, 2 ], [ 2, 2, 2, 2, 2, 2 ], [ 2, 2, 2, 2, 2, 2, 2, 2 ],
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ],
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ],
  [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ] ]
gap> List(last,Length);
[ 13, 9, 7, 4, 3, 1, 1, 0, 2, 3, 6, 8, 12, 15, 20 ]

```

**Example 8.2** (The group  $\text{Indmf}(6, 10, 1)$  is not a subgroup of the 17 irreducible maximal finite groups but is indecomposable). Let  $G = \text{Indmf}(6, 10, 1)$  be the indecomposable maximal finite group of the CARAT code  $(6, 5517, 4)$ . We will confirm that  $G$  is not a subgroup of all the 17 irreducible maximal finite groups  $\text{Imf}(6, i, j)$  of dimension 6 but is indecomposable by using the Tate cohomology  $\hat{H}^{-1}(G)$ ,  $\hat{H}^1(G)$  and  $\hat{H}^2(G)$  where  $\hat{H}^i(G) = \hat{H}^i(G, M_G)$  and  $M_G$  is the corresponding  $G$ -lattice as in Definition 1.24.

Let  $\text{Sy}_2(G)$  be a 2-Sylow subgroup of  $G$ . We have the normalizer  $N_{\text{GL}(6, \mathbb{Z})}(\text{Sy}_2(G))$  of  $\text{Sy}_2(G)$  in  $\text{GL}(6, \mathbb{Z})$  is  $\text{Sy}_2(G)$ . Suppose that there exists a group  $G'$  such that  $G < G'$  with even index. Then there exists a 2-Sylow subgroup  $\text{Sy}_2(G')$  of  $G'$  such that  $\text{Sy}_2(G) < \text{Sy}_2(G')$ . But this is impossible because  $N_{\text{GL}(6, \mathbb{Z})}(\text{Sy}_2(G)) = \text{Sy}_2(G)$ . Hence such a group  $G < G'$  with even index does not exist.

Only two groups  $G' = \text{Imf}(6, 3, 1)$  and  $G'' = \text{Imf}(6, 3, 2)$  out of the 17 groups  $\text{Imf}(6, i, j)$  may have a subgroup  $G$  with odd index. However, these two does not occur because  $\hat{H}^1(\text{Sy}_2(G)) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ ,  $\hat{H}^1(\text{Sy}_2(G')) = \mathbb{Z}/2\mathbb{Z}$  and  $\hat{H}^1(\text{Sy}_2(G'')) = \mathbb{Z}/2\mathbb{Z}$ . This implies that  $G$  is not a subgroup of all the 17 irreducible maximal finite groups  $\text{Imf}(6, i, j)$  of dimension 6.

Next, we will show that  $G$  is indecomposable. Over the field  $\mathbb{C}$  of complex numbers,  $G$  splits into 2 irreducible direct summands of degree 3. Indeed, we also see that the 6 groups of the CARAT codes  $(6, 5517, 1)$ ,  $(6, 5517, 3)$ ,  $(6, 5517, 5)$ ,  $(6, 5517, 6)$ ,  $(6, 5517, 8)$  and  $(6, 5517, 9)$  are in the same  $\mathbb{Q}$ -class of  $G$  but splits into 2 irreducible direct summands of degree 3 in  $\text{GL}(3, \mathbb{Z})$ . This implies that  $G$  splits into them in  $\text{GL}(3, \mathbb{Q})$ .

We will confirm that  $G$  is indecomposable in  $\text{GL}(6, \mathbb{Z})$ . Because of the order of  $G$  is  $2304 = 48^2$ , we see that  $G$  should be a direct product of 2 groups among 3 groups  $G_1 = \text{Imf}(3, 1, 1)$ ,  $G_2 = \text{Imf}(3, 1, 2)$  and  $G_3 = \text{Imf}(3, 1, 3)$  which are isomorphic to the group  $C_2 \times S_4$  of order 48. By computing the Tate cohomologies  $\hat{H}^i$ , we have

$$\begin{aligned}
\hat{H}^{-1}(G) &= \mathbb{Z}/2\mathbb{Z}, & \hat{H}^{-1}(G_1 \times G_1) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \hat{H}^{-1}(G_2 \times G_2) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \hat{H}^{-1}(G_3 \times G_3) &= 0, \\
\hat{H}^{-1}(G_1 \times G_2) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \hat{H}^{-1}(G_1 \times G_3) &= \mathbb{Z}/2\mathbb{Z}, & \hat{H}^{-1}(G_2 \times G_3) &= \mathbb{Z}/2\mathbb{Z}, \\
\hat{H}^1(G) &= \mathbb{Z}/2\mathbb{Z}, & \hat{H}^1(G_1 \times G_1) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \hat{H}^1(G_2 \times G_2) &= 0, & \hat{H}^1(G_3 \times G_3) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, \\
\hat{H}^1(G_1 \times G_2) &= \mathbb{Z}/2\mathbb{Z}, & \hat{H}^1(G_1 \times G_3) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \hat{H}^1(G_2 \times G_3) &= \mathbb{Z}/2\mathbb{Z}, \\
\hat{H}^2(G) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 4}, & \hat{H}^2(G_1 \times G_1) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}, & \hat{H}^2(G_2 \times G_2) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \hat{H}^2(G_3 \times G_3) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}, \\
\hat{H}^2(G_1 \times G_2) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 5}, & \hat{H}^2(G_1 \times G_3) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}, & \hat{H}^2(G_2 \times G_3) &= (\mathbb{Z}/2\mathbb{Z})^{\oplus 5}.
\end{aligned}$$

Hence  $G$  is indecomposable in  $\text{GL}(6, \mathbb{Z})$ .

```

gap> Read("caratnumber.gap");
gap> Read("FlabbyResolution.gap");
gap> Read("Hn.gap");
gap> Read("KS.gap");

gap> G:=CaratMatGroupZClass(6,5517,4);; # G=Indmf(6,10,1)
gap> Order(G); # #G=2304=48^2=2^8*3^2
2304
gap> imf6:=AllImfMatrixGroups(6);
[ ImfMatrixGroup(6,1,1), ImfMatrixGroup(6,1,2), ImfMatrixGroup(6,1,3),
  ImfMatrixGroup(6,2,1), ImfMatrixGroup(6,3,1), ImfMatrixGroup(6,3,2),

```

```

ImfMatrixGroup(6,4,1), ImfMatrixGroup(6,4,2), ImfMatrixGroup(6,5,1),
ImfMatrixGroup(6,6,1), ImfMatrixGroup(6,6,2), ImfMatrixGroup(6,6,3),
ImfMatrixGroup(6,7,1), ImfMatrixGroup(6,7,2), ImfMatrixGroup(6,8,1),
ImfMatrixGroup(6,9,1), ImfMatrixGroup(6,9,2) ]
gap> List(Imf6,x->Order(x)/2304);
[ 20, 20, 20, 9/2, 45, 45, 35/8, 35/8, 7/24, 5/48, 5/48, 5/48, 2, 2, 10, 1/8, 1/8 ]
gap> G2:=SylowSubgroup(G,2); # G2 is a 2-sylow group of G
<group of 6x6 matrices of size 256 in characteristic 0>
gap> Normalizer(GL(6,Integers),G2);
<matrix group with 14 generators>
gap> Order(last); # (Normalizer of G2)=G2
256
gap> H1(G2);
[ 1, 1, 1, 1, 2, 2 ]
gap> H1(SylowSubgroup(ImfMatrixGroup(6,3,1),2));
[ 1, 1, 1, 1, 1, 2 ]
gap> H1(SylowSubgroup(ImfMatrixGroup(6,3,2),2));
[ 1, 1, 1, 1, 1, 2 ]

gap> Set(AllDirectProductIndmfMatrixGroups([3,3]),CaratZClass);
[ [ 6, 5517, 1 ], [ 6, 5517, 3 ], [ 6, 5517, 5 ],
  [ 6, 5517, 6 ], [ 6, 5517, 8 ], [ 6, 5517, 9 ] ]

gap> ig:=IrreducibleRepresentations(G);;
gap> List(ig,x->Sum(G,y->Trace(y)*Trace(Image(x,y))));
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 2304, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 2304, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
gap> Filtered([1..Length(ig)],x->last[x]>0);
[ 44, 65 ]
gap> ig[44];
CompositionMapping( [ (2,6)(5,8)(7,11), (5,8), (1,12)(2,6)(4,5,9,8)(7,11), (6,7),
  (1,2,6)(3,4,8)(5,10,9)(7,12,11), (1,2,6)(7,12,11), (4,9)(5,8), (1,12)(6,7), (2,11)(6,7),
  (3,10)(4,9) ] ->
[ [ [ -1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ], [ [ -1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
  [ [ 0, 1, 0 ], [ -1, 0, 0 ], [ 0, 0, 1 ] ], [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
  [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ], [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
  [ [ -1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, 1 ] ], [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
  [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ], [ [ 1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, -1 ] ] ],
  <action isomorphism> )
gap> ig[65];
CompositionMapping( [ (2,6)(5,8)(7,11), (5,8), (1,12)(2,6)(4,5,9,8)(7,11), (6,7),
  (1,2,6)(3,4,8)(5,10,9)(7,12,11), (1,2,6)(7,12,11), (4,9)(5,8), (1,12)(6,7), (2,11)(6,7),
  (3,10)(4,9) ] ->
[ [ [ 1, 0, 0 ], [ 0, 0, 1 ], [ 0, 1, 0 ] ], [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],
  [ [ -1, 0, 0 ], [ 0, 0, 1 ], [ 0, 1, 0 ] ], [ [ 1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, 1 ] ],
  [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ], [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ],
  [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ], [ [ -1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, 1 ] ],
  [ [ 1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, -1 ] ], [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ] ],
  <action isomorphism> )
gap> Imf3:=AllImfMatrixGroups(3);
[ ImfMatrixGroup(3,1,1), ImfMatrixGroup(3,1,2), ImfMatrixGroup(3,1,3) ]
gap> List(Imf3,Size);

```

```

[ 48, 48, 48 ]
gap> G11:=DirectProductMatrixGroup([Imf3[1],Imf3[1]]);;
gap> G12:=DirectProductMatrixGroup([Imf3[1],Imf3[2]]);;
gap> G13:=DirectProductMatrixGroup([Imf3[1],Imf3[3]]);;
gap> G22:=DirectProductMatrixGroup([Imf3[2],Imf3[2]]);;
gap> G23:=DirectProductMatrixGroup([Imf3[2],Imf3[3]]);;
gap> G33:=DirectProductMatrixGroup([Imf3[3],Imf3[3]]);;
gap> List([G11,G22,G33,G12,G13,G23],CaratZClass);
[ [ 6, 5517, 1 ], [ 6, 5517, 9 ], [ 6, 5517, 6 ],
  [ 6, 5517, 5 ], [ 6, 5517, 3 ], [ 6, 5517, 8 ] ]
gap> List([G,G11,G22,G33,G12,G13,G23],x->TateCohomology(x,-1));
[ [ 2 ], [ 2, 2 ], [ 2, 2 ], [ ], [ 2, 2 ], [ 2 ], [ 2 ] ]
gap> List([G,G11,G22,G33,G12,G13,G23],x->TateCohomology(x,1));
[ [ 2 ], [ 2, 2 ], [ ], [ 2, 2 ], [ 2 ], [ 2, 2 ], [ 2 ] ]
gap> List([G,G11,G22,G33,G12,G13,G23],x->TateCohomology(x,2));
[ [ 2, 2, 2, 2 ], [ 2, 2, 2, 2, 2, 2, 2, 2 ], [ 2, 2 ], [ 2, 2, 2, 2, 2, 2, 2, 2 ],
  [ 2, 2, 2, 2, 2 ], [ 2, 2, 2, 2, 2, 2, 2, 2 ], [ 2, 2, 2, 2, 2 ] ]
gap> List(last,Length);
[ 4, 8, 2, 8, 5, 8, 5 ]

```

## 9. PROOF OF THEOREM 1.25

Let  $G$  be a finite subgroup of  $\mathrm{GL}(4, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank 4 as in Definition 1.24.

**Step 1.** The case where  $M_G$  is decomposable.

We assume that  $M_G \simeq M_1 \oplus M_2$  is decomposable with  $\mathrm{rank} M_1 \geq \mathrm{rank} M_2 \geq 1$ . Thus  $\mathrm{rank} M_1 \leq 3$  and  $\mathrm{rank} M_2 \leq 2$ . By Theorem 1.1, we have  $[M_2]^{fl} = 0$ . It follows from Lemma 2.14 that  $[M_G]^{fl} = [M_1]^{fl}$ . By Theorem 1.2 and Lemma 2.14,  $[M_1]^{fl} \neq 0 \iff [M_1]^{fl}$  is not invertible  $\iff G/N_1$  is conjugate to one of the 15 groups as in Table 1 where  $N_1 = \{\sigma \in G \mid \sigma(v) = v \text{ for any } v \in M_1\}$ . There exist exactly 64 such  $G$ -lattices  $M_G \simeq M_1 \oplus M_2$  with  $[M_G]^{fl} \neq 0$  (equivalently  $[M_G]^{fl}$  is not invertible in this case) whose GAP codes  $\mathcal{N}_{31}$  are computed in Example 4.12 (see also Table 3).

**Step 2.**  $[M_G]^{fl} = 0$  for  $G = \mathrm{Imf}(4, 2, 1)$ ,  $\mathrm{Imf}(4, 3, 1)$  and  $\mathrm{Imf}(4, 4, 1)$ .

We assume that  $M_G$  is indecomposable. There exist 295 indecomposable  $G$ -lattices  $M_G$  of rank 4 (see Example 4.9). Let  $\mathrm{Imf}(4, i, j) \leq \mathrm{GL}(4, \mathbb{Z})$  be the  $j$ -th  $\mathbb{Z}$ -class of the  $i$ -th  $\mathbb{Q}$ -class of the irreducible maximal finite group of dimension 4 which corresponds to the GAP command `ImfMatrixGroup(4,i,j)`. There exist exactly 6 such maximal groups  $\mathrm{Imf}(4, 1, 1)$ ,  $\mathrm{Imf}(4, 2, 1)$ ,  $\mathrm{Imf}(4, 3, 1)$ ,  $\mathrm{Imf}(4, 3, 2)$ ,  $\mathrm{Imf}(4, 4, 1)$  and  $\mathrm{Imf}(4, 5, 1)$  of order 1152, 288, 240, 240, 384 and 144 respectively. They also form the maximal indecomposable finite groups of dimension 4 (see Subsection 4.0).

By using `flf1` in Algorithm F3, we get  $[M_G]^{fl} = 0$  for  $G = \mathrm{Imf}(4, 4, 1)$  (see Example 5.2).

We may also verify that  $[M_G]^{fl} = 0$  for  $G = \mathrm{Imf}(4, 2, 1)$  and  $\mathrm{Imf}(4, 3, 1)$  by using `StablyPermutationFCheckP` in Algorithm F6: Method II and `StablyPermutationFCheck` in Algorithm F5: Method I (1) respectively (see Example 5.7 and Example 5.5). Hence, by Lemma 2.17,  $[M_H]^{fl} = 0$  for any subgroup  $H$  of  $\mathrm{Imf}(4, 2, 1)$ ,  $\mathrm{Imf}(4, 3, 1)$  and  $\mathrm{Imf}(4, 4, 1)$ .

**Step 3.** Determination of all  $G$ -lattices  $M_G$  whose flabby class  $[M_G]^{fl}$  is not invertible.

Assume that  $M_G$  is indecomposable. By Step 2,  $[M_H]^{fl}$  is invertible for any subgroups  $H$  of  $\mathrm{Imf}(4, 2, 1)$ ,  $\mathrm{Imf}(4, 3, 1)$  and  $\mathrm{Imf}(4, 4, 1)$ . We will check whether  $[M_H]^{fl}$  is invertible for subgroups  $H$  of  $\mathrm{Imf}(4, 1, 1)$ ,  $\mathrm{Imf}(4, 3, 2)$  and  $\mathrm{Imf}(4, 5, 1)$ .

There exist 182, 36 and 50 indecomposable  $G$ -lattices  $M_G$  where  $G$  is (a conjugacy class of) a subgroup of  $\mathrm{Imf}(4, 1, 1)$ ,  $\mathrm{Imf}(4, 3, 2)$  and  $\mathrm{Imf}(4, 5, 1)$  respectively. Some of them are also subgroups of  $\mathrm{Imf}(4, 2, 1)$ ,  $\mathrm{Imf}(4, 3, 1)$

or  $\text{Imf}(4, 4, 1)$  as in Step 2. Then we should treat 166, 8 and 27 indecomposable  $G$ -lattices  $M_G$  where  $G$  is a subgroup of  $\text{Imf}(4, 1, 1)$ ,  $\text{Imf}(4, 3, 2)$  and  $\text{Imf}(4, 5, 1)$  respectively but not a subgroup of  $\text{Imf}(4, 2, 1)$ ,  $\text{Imf}(4, 3, 1)$  and  $\text{Imf}(4, 4, 1)$ . We denote the set of the corresponding GAP codes of  $M_G$  in each case by  $\tilde{\mathcal{U}}_{411}$ ,  $\tilde{\mathcal{U}}_{432}$  and  $\tilde{\mathcal{U}}_{451}$ , i.e.  $\#\tilde{\mathcal{U}}_{411} = 166$ ,  $\#\tilde{\mathcal{U}}_{432} = 8$ ,  $\#\tilde{\mathcal{U}}_{451} = 27$ , (see Example 9.1 below).

By using `IsInvertibleF` in Algorithm F2, we may determine all of the  $G$ -lattices  $M_G$  whose flabby class  $[M_G]^{fl}$  is not invertible. There exist 137, 0, 27  $G$ -lattices  $M_G$  with  $[M_G]^{fl}$  not invertible where the GAP code of  $G$  is  $l \in \tilde{\mathcal{U}}_{411}$ ,  $\tilde{\mathcal{U}}_{432}$ ,  $\tilde{\mathcal{U}}_{451}$  respectively. Note that  $[M_G]^{fl}$  is invertible for  $G = \text{Imf}(4, 3, 2)$ . Because the 12  $G$ -lattices of them are overlapped, there exist 152  $G$ -lattices  $M_G$  with  $[M_G]^{fl}$  not invertible (see Example 9.1 below and Table 4).

**Step 4.** Determination whether  $[M_G]^{fl} = 0$ .

We should determine whether  $[M_G]^{fl} = 0$  for the remaining cases  $\mathcal{U}_{411}$  and  $\mathcal{U}_{432}$  of  $\tilde{\mathcal{U}}_{411}$  and  $\tilde{\mathcal{U}}_{432}$  as in Step 3. We see that  $\#\mathcal{U}_{411} = 29$ ,  $\#\mathcal{U}_{432} = 8$  and  $\mathcal{U}_{411}$  and  $\mathcal{U}_{432}$  are given by

$$\begin{aligned} \mathcal{U}_{411} = & \{(4, 5, 1, 7), (4, 5, 1, 11), (4, 5, 1, 13), (4, 6, 1, 10), (4, 6, 1, 12), (4, 6, 2, 11), (4, 12, 1, 6), (4, 12, 1, 7), \\ & (4, 12, 3, 10), (4, 12, 3, 12), (4, 12, 3, 13), (4, 12, 4, 13), (4, 13, 1, 6), (4, 13, 2, 6), (4, 13, 3, 6), (4, 13, 4, 6), \\ & (4, 13, 6, 6), (4, 13, 7, 12), (4, 24, 1, 4), (4, 24, 1, 6), (4, 24, 3, 4), (4, 24, 3, 6), (4, 24, 4, 6), (4, 25, 1, 5), \\ & (4, 25, 3, 5), (4, 25, 4, 5), (4, 25, 7, 5), (4, 25, 8, 5), (4, 33, 2, 1)\}, \\ \mathcal{U}_{432} = & \{(4, 31, 1, 3), (4, 31, 1, 4), (4, 31, 2, 2), (4, 31, 3, 2), (4, 31, 4, 2), (4, 31, 5, 2), (4, 31, 6, 2), (4, 31, 7, 2)\}. \end{aligned}$$

We will show that there exist exactly 7 groups  $G$  of the GAP codes  $(4, 33, 2, 1) \in \mathcal{U}_{411}$  and  $(4, 31, 1, 3), (4, 31, 1, 4), (4, 31, 2, 2), (4, 31, 4, 2), (4, 31, 5, 2)$  and  $(4, 31, 7, 2) \in \mathcal{U}_{432}$  such that  $[M_G]^{fl} \neq 0$ .

First we treat the case of  $\mathcal{U}_{411}$ . Each group  $G$  of the GAP code  $l \in \mathcal{U}_{411}$  is contained in at least one of the groups of the 21st, 27th, 28th and 29th GAP codes, i.e.  $(4, 24, 3, 4)$ ,  $(4, 25, 7, 5)$ ,  $(4, 25, 8, 5)$  and  $(4, 33, 2, 1)$ , in  $\mathcal{U}_{411}$  (see Example 9.2 below). By using `flf1` in Algorithm F3, (resp. `StablyPermutationFCheckP` in Algorithm F6: Method II), we may confirm that  $[M_G]^{fl} = 0$  for the 21st and 27th (resp. 28th) groups  $G$  of the GAP code  $(4, 24, 3, 4)$  and  $(4, 25, 7, 5)$  (resp.  $(4, 25, 8, 5)$ ) (see Example 9.2 below). None of the groups  $G$  of the GAP code  $l \in \mathcal{U}_{411}$  is contained in the 29th group  $G$  other than itself. By Lemma 2.17,  $[M_G]^{fl} = 0$  for all groups  $G$  of the GAP code  $l \in \mathcal{U}_{411}$  except for the 29th group  $G$  of the GAP code  $(4, 33, 2, 1)$ . By using `PossibilityOfStablyPermutationF` in Algorithm F5, we may see that  $[M_G]^{fl} \neq 0$  for the 29th group  $G$  (see Example 9.2 below).

Next we will consider the case of  $\mathcal{U}_{432}$ . By using `PossibilityOfStablyPermutationF` in Algorithm F5, we see that  $[M_G]^{fl} \neq 0$  for the 1st and the 2nd groups  $G$  of the GAP codes  $(4, 31, 1, 3)$  and  $(4, 31, 1, 4)$  (see Example 5.4 and Example 9.3 below). Because the 3rd, 5th, 6th and 8th groups  $G$  of the GAP codes  $(4, 31, 2, 2)$ ,  $(4, 31, 4, 2)$ ,  $(4, 31, 5, 2)$  and  $(4, 31, 7, 2)$  contain the 1st or 2nd ones, it follows from Lemma 2.17 that  $[M_G]^{fl} \neq 0$ . By using `StablyPermutationFCheckP` in Algorithm F6: Method II, we may confirm that  $[M_G]^{fl} = 0$  for the 4th and 7th groups  $G$  of the GAP codes  $(4, 31, 3, 2)$  and  $(4, 31, 6, 2)$  (see Example 9.3 below).

**Step 5.**  $[M_G]^{fl} = 0$  if and only if  $[M_G]^{fl}$  is of finite order in  $C(G)/S(G)$ .

We should show that if  $[M_G]^{fl} \neq 0$ , then  $[M_G]^{fl}$  is of infinite order in  $C(G)/S(G)$ . Note that  $[M_G]^{fl}$  is of finite order if and only if there exist permutation  $G$ -lattices  $P, P'$  such that  $[(M_G)^{\oplus r}]^{fl} \oplus P \simeq P'$  for some  $r \geq 1$ . Thus if  $[M_G]^{fl}$  is not invertible, then  $[M_G]^{fl}$  is of infinite order. Hence it is enough to verify that  $[(M_G)^{\oplus r}]^{fl} \neq 0$  for the 7 cases as in Table 2, i.e. the 7 cases where  $[M_G]^{fl}$  is not zero but invertible. By Step 4, we should show that  $[(M_G)^{\oplus r}]^{fl} \neq 0$  for only the 3 groups  $G$  of the GAP codes  $(4, 31, 1, 3)$ ,  $(4, 31, 1, 4)$  and  $(4, 33, 2, 1)$  because the remaining 4 groups contain the 1st or the 2nd one.

By using `PossibilityOfStablyPermutationF` as in Algorithm F4, for 2 groups  $G \simeq F_{20}$  of the GAP codes  $(4, 31, 1, 3)$  and  $(4, 31, 1, 4)$ , we get the flabby class  $[F] = [M_G]^{fl}$  with rank 16 and see that only the possibility of the isomorphism (10) is

$$(13) \quad \mathbb{Z}[G] \oplus \mathbb{Z}[G/C_2] \oplus \mathbb{Z}[G/C_5] \simeq \mathbb{Z}[G/D_5] \oplus 2F$$

(the rank of the both sides is  $20 + 10 + 4 = 2 + 2 \times 16 = 34$ ). By using the algorithm `TateCohomology` as in Section 8, however, for the group  $G$  of the GAP code  $(4, 31, 1, 3)$ , we see that the 2nd (Tate) cohomologies of the both sides are  $\hat{H}^2(G_1, M_{G_1}) = \mathbb{Z}/2\mathbb{Z}$  and  $\hat{H}^2(G_2, M_{G_2}) = \mathbb{Z}/10\mathbb{Z}$  where  $G_1$  (resp.  $G_2$ ) is the matrix representation group of the action of  $G$  on the left (resp. right) hand side of (13). Thus the isomorphism (13) is

impossible. Hence  $[F^{\oplus r}] \neq 0$  for any  $r \geq 1$ . Similarly, for the group  $G$  of the GAP code  $(4, 31, 1, 4)$ , we see that  $\widehat{H}^2(G_1, M_{G_1}) = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$  and  $\widehat{H}^2(G_2, M_{G_2}) = \mathbb{Z}/10\mathbb{Z}$  (see Example 9.4 below).

For the group  $G \simeq C_3 \rtimes C_8$  of the GAP code  $(4, 33, 2, 1)$ , by using `PossibilityOfStablyPermutationF` as in Algorithm F4, only the possibility of (10) is

$$(14) \quad \mathbb{Z}[G] \oplus \mathbb{Z}[G/C_2] \oplus \mathbb{Z}[G/C_3] \simeq \mathbb{Z}[G/C_6] \oplus 2F$$

where  $F$  is of rank 20 with  $[F] = [M_G]^{fl}$  (the rank of the both sides is  $24 + 12 + 8 = 4 + 2 \times 20 = 44$ ). However, the Tate cohomologies of the both sides  $\widehat{H}^n(G_1, M_{G_1}) \simeq \widehat{H}^n(G_2, M_{G_2})$  coincide for  $-7 \leq n \leq 7$  (see Example 9.4 below). Thus we need another observation.

Assume that the isomorphism (14) holds. Let  $G_1$  (resp.  $G_2$ ) be the matrix representation group of the action of  $G$  on the left (resp. right) hand side of (14). By using `StablyPermutationFCheckGen` as in Algorithm F6: Method II, we may obtain  $G_1$  and  $G_2$  of rank 44 (see Example 9.4 below). We choose the generators  $a, b$  of  $G$  such that  $G = \langle a, b \mid a^3 = b^8 = 1, b^{-1}ab = a^{-1} \rangle \simeq \langle a \rangle \rtimes \langle b \rangle$ . We take the reduction  $\overline{G}_1 = \langle \overline{a}_1, \overline{b}_1 \rangle$ ,  $\overline{G}_2 = \langle \overline{a}_2, \overline{b}_2 \rangle \leq \text{GL}(44, \mathbb{F}_3)$  over the field  $\mathbb{F}_3$  of 3 elements. Because the subgroup  $\langle a \rangle \simeq C_3$  is normal in  $G$ ,  $v_i = \{(\overline{a}_i - \overline{I}_{44})^2 \mid a_i \in G_i\}$  is  $\overline{G}_i$ -invariant ( $i = 1, 2$ ). Consider the action of  $b$  on the  $\overline{G}_i$ -invariant space  $v_i$  over the field  $\mathbb{F}_9$  of 9 elements and compare the dimensions of each eigen spaces of the action matrix of  $b$  on  $v_i$  ( $i = 1, 2$ ) (we see  $\dim(v_1) = \dim(v_2) = 12$ ). Then we get the (multi-)set of such dimensions  $\{2, 2, 2, 2, 2, 0, 2, 0\}$  and  $\{2, 2, 1, 2, 1, 1, 2, 1\}$  for  $v_1$  and  $v_2$  respectively (see Example 9.4 below). This yields a contradiction. Hence the isomorphism (14) is impossible.

**Step 6.**  $[M_G]^{fl} = -[J_{S_5/S_4}]^{fl}$  for the group  $G \simeq S_5$  of the GAP code  $(4, 32, 5, 2)$ .

Let  $G_1, G_2 \simeq S_5 \leq \text{GL}(4, \mathbb{Z})$  be the groups of the GAP codes  $(4, 31, 4, 2)$  and  $(4, 31, 5, 2)$ . Note that  $M_{G_1} \simeq J_{S_5/S_4}$ . Let  $M_G \simeq M_{G_1} \oplus M_{G_2}$  be the  $S_5$ -lattice of rank 8. By Step 4,  $[M_{G_1}]^{fl}$  and  $[M_{G_2}]^{fl}$  is not zero but invertible, and we will show that  $[M_G]^{fl} = [F] = 0$ .

By using `StablyPermutationFCheckGen` as in Algorithm F6: Method II, it is possible that

$$(15) \quad \mathbb{Z}[G/H_4] \oplus \mathbb{Z}[G/H_{13}] \oplus F \simeq \mathbb{Z}[G/H_8] \oplus \mathbb{Z}[G/H_9] \oplus \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{11}]$$

where  $H_4 \simeq C_3$ ,  $H_{13} \simeq D_5$ ,  $H_8 \simeq C_5$ ,  $H_9 \simeq C_6$ ,  $H_{10} \simeq H_{11} \simeq S_3$  (the rank of the both sides is  $40 + 12 + 32 = 24 + 20 + 20 + 20 = 84$ ). However, we could not establish the isomorphism (15).

As in Example 5.6 (Algorithm F5: Method I (2)), after adding  $\mathbb{Z}$  to the both sides of (15), we may confirm that the isomorphism

$$\mathbb{Z}[G/C_3] \oplus \mathbb{Z}[G/D_5] \oplus F \oplus \mathbb{Z} \simeq \mathbb{Z}[G/C_5] \oplus \mathbb{Z}[G/C_6] \oplus \mathbb{Z}[G/S_3^{(1)}] \oplus \mathbb{Z}[G/S_3^{(2)}] \oplus \mathbb{Z}$$

holds (see Example 9.5 below)

**Step 7.**  $[M_G]^{fl} = -[J_{F_{20}/C_4}]^{fl}$  for the group  $G \simeq F_{20}$  of the GAP code  $(4, 31, 1, 4)$ .

Let  $G'_1, G'_2 \simeq F_{20} \leq \text{GL}(4, \mathbb{Z})$  be the groups of the GAP codes  $(4, 31, 1, 3)$  and  $(4, 31, 1, 4)$ . Note that  $M_{G'_1} \simeq J_{F_{20}/C_4}$ . Let  $M_G \simeq M_{G'_1} \oplus M_{G'_2}$  be the  $F_{20}$ -lattice of rank 8. By Step 4,  $[M_{G'_1}]^{fl}$  and  $[M_{G'_2}]^{fl}$  is not zero but invertible. We should show that  $[M_G]^{fl} = [F] = 0$ . Because  $G'_1$  (resp.  $G'_2$ ) is a subgroup of  $G_1$  (resp.  $G_2$ ) where  $G_1$  (resp.  $G_2$ ) is the group  $S_5$  as in Step 6, the assertion follows from Step 6 and Lemma 2.17.

We will give another (direct) proof below. By using `StablyPermutationFCheckGen` as in Algorithm F6: Method II, it is possible that

$$(16) \quad \mathbb{Z}[G] \oplus \mathbb{Z}[G/C_2] \oplus \mathbb{Z}[G/C_5] \simeq \mathbb{Z}[G/D_5] \oplus F$$

(the rank of the both sides is  $20 + 10 + 4 = 2 + 32 = 34$ ). And, after some efforts, we may confirm that the isomorphism (16) actually holds (see Example 9.6 below).  $\square$

**Example 9.1** (Determination whether  $[M_G]^{fl}$  is invertible).

```
gap> Read("crystcat.gap");
gap> Read("caratnumber.gap");
gap> Read("FlabbyResolution.gap");
gap> Read("KS.gap");
```

```

gap> imf411sub:=Set(ConjugacyClassesSubgroups2(ImfMatrixGroup(4,1,1)),
> x->CrystCatZClass(Representative(x)));;
gap> imf421sub:=Set(ConjugacyClassesSubgroups2(ImfMatrixGroup(4,2,1)),
> x->CrystCatZClass(Representative(x)));;
gap> imf431sub:=Set(ConjugacyClassesSubgroups2(ImfMatrixGroup(4,3,1)),
> x->CrystCatZClass(Representative(x)));;
gap> imf432sub:=Set(ConjugacyClassesSubgroups2(ImfMatrixGroup(4,3,2)),
> x->CrystCatZClass(Representative(x)));;
gap> imf441sub:=Set(ConjugacyClassesSubgroups2(ImfMatrixGroup(4,4,1)),
> x->CrystCatZClass(Representative(x)));;
gap> imf451sub:=Set(ConjugacyClassesSubgroups2(ImfMatrixGroup(4,5,1)),
> x->CrystCatZClass(Representative(x)));;

gap> ind4:=LatticeDecompositions(4)[NrPartitions(4)];;
gap> imf411ind:=Intersection(imf411sub,ind4);;
gap> imf421ind:=Intersection(imf421sub,ind4);;
gap> imf431ind:=Intersection(imf431sub,ind4);;
gap> imf432ind:=Intersection(imf432sub,ind4);;
gap> imf441ind:=Intersection(imf441sub,ind4);;
gap> imf451ind:=Intersection(imf451sub,ind4);;
gap> List([imf411ind,imf421ind,imf431ind,imf432ind,imf441ind,imf451ind],Length)
[ 182, 50, 36, 36, 45, 50 ]

gap> U411t:=Difference(imf411ind,Union([imf421ind,imf431ind,imf441ind]));;
gap> U432t:=Difference(imf432ind,Union([imf421ind,imf431ind,imf441ind]));;
gap> U451t:=Difference(imf451ind,Union([imf421ind,imf431ind,imf441ind]));;
gap> List([U411t,U432t,U451t],Length);
[ 166, 8, 27 ]

gap> imf411n:=Filtered(U411t,x->IsInvertibleF(MatGroupZClass(x[1],x[2],x[3],x[4]))=false);;
gap> imf432n:=Filtered(U432t,x->IsInvertibleF(MatGroupZClass(x[1],x[2],x[3],x[4]))=false);;
gap> imf451n:=Filtered(U451t,x->IsInvertibleF(MatGroupZClass(x[1],x[2],x[3],x[4]))=false);;
gap> List([imf411n,imf432n,imf451n],Length);
[ 137, 0, 27 ]

gap> N4:=Union(imf411n,imf451n);;
gap> Length(N4);
152
gap> Intersection(imf411n,imf451n);
[ [ 4, 22, 1, 1 ], [ 4, 22, 2, 1 ], [ 4, 22, 3, 1 ], [ 4, 22, 4, 1 ],
  [ 4, 22, 5, 1 ], [ 4, 22, 5, 2 ], [ 4, 22, 6, 1 ], [ 4, 22, 7, 1 ],
  [ 4, 22, 8, 1 ], [ 4, 22, 9, 1 ], [ 4, 22, 10, 1 ], [ 4, 22, 11, 1 ] ]
gap> Length(last);
12

gap> U411:=Difference(U411t,imf411n);
[ [ 4, 5, 1, 7 ], [ 4, 5, 1, 11 ], [ 4, 5, 1, 13 ], [ 4, 6, 1, 10 ], [ 4, 6, 1, 12 ],
  [ 4, 6, 2, 11 ], [ 4, 12, 1, 6 ], [ 4, 12, 1, 7 ], [ 4, 12, 3, 10 ], [ 4, 12, 3, 12 ],
  [ 4, 12, 3, 13 ], [ 4, 12, 4, 13 ], [ 4, 13, 1, 6 ], [ 4, 13, 2, 6 ], [ 4, 13, 3, 6 ],
  [ 4, 13, 4, 6 ], [ 4, 13, 6, 6 ], [ 4, 13, 7, 12 ], [ 4, 24, 1, 4 ], [ 4, 24, 1, 6 ],
  [ 4, 24, 3, 4 ], [ 4, 24, 3, 6 ], [ 4, 24, 4, 6 ], [ 4, 25, 1, 5 ], [ 4, 25, 3, 5 ],
  [ 4, 25, 4, 5 ], [ 4, 25, 7, 5 ], [ 4, 25, 8, 5 ], [ 4, 33, 2, 1 ] ]
gap> U432:=Difference(U432t,imf432n);
[ [ 4, 31, 1, 3 ], [ 4, 31, 1, 4 ], [ 4, 31, 2, 2 ], [ 4, 31, 3, 2 ],
  [ 4, 31, 4, 2 ], [ 4, 31, 5, 2 ], [ 4, 31, 6, 2 ], [ 4, 31, 7, 2 ] ]

```

```
gap> U451:=Difference(U451t,imf451n);
[ ]
gap> List([U411,U432,U451],Length);
[ 29, 8, 0 ]
```

**Example 9.2** (Determination whether  $[M_G]^{f_l} = 0$  for the GAP code  $l \in \mathcal{U}_{411}$ ).

```
gap> U411g:=List(U411,x->MatGroupZClass(x[1],x[2],x[3],x[4]));;
gap> U411sub:=List(U411g,x->Set(ConjugacyClassesSubgroups2(x),
> y->CrystCatZClass(Representative(y))));;
gap> U411inc:=List(U411sub,x->Set(Intersection(x,U411),y->Position(U411,y)));
[ [ 1 ], [ 2 ], [ 3 ], [ 2, 4 ], [ 3, 5 ], [ 2, 6 ], [ 7 ], [ 8 ], [ 1, 7, 9 ],
  [ 2, 8, 10 ], [ 3, 8, 11 ], [ 3, 8, 12 ], [ 8, 13 ], [ 8, 14 ], [ 2, 3, 15 ],
  [ 2, 3, 16 ], [ 2, 3, 4, 5, 8, 10, 11, 13, 16, 17 ],
  [ 2, 3, 5, 6, 8, 10, 12, 14, 15, 18 ], [ 1, 19 ], [ 3, 20 ],
  [ 1, 7, 9, 19, 21 ], [ 3, 8, 11, 20, 22 ], [ 3, 8, 12, 20, 23 ],
  [ 3, 5, 20, 24 ], [ 2, 3, 16, 20, 25 ], [ 2, 3, 15, 20, 26 ],
  [ 2, 3, 4, 5, 8, 10, 11, 13, 16, 17, 20, 22, 24, 25, 27 ],
  [ 2, 3, 5, 6, 8, 10, 12, 14, 15, 18, 20, 23, 24, 26, 28 ], [ 29 ] ]

gap> U411[21]; # checking  $[M]^{\{f_l\}}=0$ 
[ 4, 24, 3, 4 ]
gap> G:=MatGroupZClass(4,24,3,4);; # G=S4
gap> flfl(G);
[ ]

gap> U411[27]; # checking  $[M]^{\{f_l\}}=0$ 
[ 4, 25, 7, 5 ]
gap> G:=MatGroupZClass(4,25,7,5);; # G=C2xS4
gap> flfl(G);
[ ]

gap> U411[28]; # checking  $[M]^{\{f_l\}}=0$ 
[ 4, 25, 8, 5 ]
gap> G:=MatGroupZClass(4,25,8,5);; # G=C2xS4
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 10
10

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),5);;
gap> Set(List(mis,Length))-4; # Method III could not apply
[ 10, 16, 18, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 44, 46, 50 ]

gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 0, 0, 0, 0, -2, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 1, 1, 0, -1, 0,
  0, 0, 0, 1, 1, -1, 1, -1, -1 ],
  [ 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, -1, 0, 0, 0, 1, 0, 0, 0, 0,
  0, 0, 2, 1, 1, 0, 0, -1, -1 ],
  [ 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 1, 0, -1, 0,
  0, 0, 0, 0, 0, -1, 1, 0, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, -1, 0,
  0, -1, 0, 1, 1, 1, 1, -1, -1 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0,
  0, 0, 0, -1, 0, 0, 0, 1, -1 ] ]
gap> l:=ll[Length(ll)];
```



```

[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
  1, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, -1 ]
gap> bp:=StablyPermutationFCheckP(G,Nlist(1),Plist(1));;
gap> Length(bp);
20
gap> Length(bp[1]); # rank of the both sides of (10) is 13
13
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..1000],x->List([1..20],y->Random(rs,[0,1]))));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1);
[ [ 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0 ],
  [ 0, 1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1 ] ]
gap> p:=last[1]*bp;
[ [ 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1 ],
  [ 0, 0, 1, 0, 0, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 2, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 ],
  [ 1, 0, 0, 0, 2, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 0, 1, 0, 0 ],
  [ 0, 0, 2, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 2, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
  [ 1, 0, 0, 0, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],
  [ 2, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ] ]
gap> Determinant(p);
1
gap> StablyPermutationFCheckMat(G,Nlist(1),Plist(1),p);
true

gap> U411[29]; # checking [M]^{f1}: non-zero
[ 4, 33, 2, 1 ]
gap> G:=MatGroupZClass(4,33,2,1);; # G=C3:C8
gap> PossibilityOfStablyPermutationF(G);
[ [ 1, 1, 1, 0, -1, 0, 0, 0, -2 ] ]

gap> i411:=[U411[29]];
[ [ 4, 33, 2, 1 ] ]

```

**Example 9.3** (Determination whether  $[M_G]^{fl} = 0$  for the GAP code  $l \in \mathcal{U}_{432}$ ).

```

gap> U432g:=List(U432,x->MatGroupZClass(x[1],x[2],x[3],x[4]));;
gap> U432sub:=List(U432g,x->Set(ConjugacyClassesSubgroups2(x),
> y->CrystCatZClass(Representative(y))));;
gap> U432inc:=List(U432sub,x->Set(Intersection(x,U432),y->Position(U432,y)));
[ [ 1 ], [ 2 ], [ 1, 2, 3 ], [ 4 ], [ 1, 4, 5 ], [ 2, 4, 6 ], [ 4, 7 ],
  [ 1, 2, 3, 4, 5, 6, 7, 8 ] ]

gap> U432[1]; # checking [M]^{f1}: non-zero
[ 4, 31, 1, 3 ]
gap> G:=MatGroupZClass(4,31,1,3);; # G=F20
gap> PossibilityOfStablyPermutationF(G);

```

```

[ [ 1, 1, 0, 1, -1, 0, -2 ] ]

gap> U432[2]; # checking [M]^{fl}: non-zero
[ 4, 31, 1, 4 ]
gap> G:=MatGroupZClass(4,31,1,4);; # G=F20
gap> PossibilityOfStablyPermutationF(G);
[ [ 1, 1, 0, 1, -1, 0, -2 ] ]

gap> U432[4]; # checking [M]^{fl}=0
[ 4, 31, 3, 2 ]
gap> G:=MatGroupZClass(4,31,3,2);; # G=A5
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 16
16

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),5);;
gap> Set(List(mis,Length))-4; # Method III could not apply
[ 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81 ]

gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, -2, -1, 0, 0, 1, 1, 1, -1, 0 ], [ 0, 0, 0, 0, 1, 1, -1, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 1, 1, -1, 0, 0, -1 ]
gap> bp:=StablyPermutationFCheckP(G,Nlist(l),Plist(l));;
gap> Length(bp);
11
gap> Length(bp[1]); # rank of the both sides of (10) is 22
22
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..100000],x->List([1..11],y->Random(rs,[-1..2])));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1);
[ [ 2, 0, 2, -1, 0, 0, 1, 0, 0, 1, 1 ], [ 2, 0, 0, 1, -1, 1, 0, 1, 0, 1, 1 ] ]
gap> p:=last[1]*bp;
[ [ 2, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 2, -1, 2, 2, 2, -1, 2, -1, -1, -1 ],
  [ 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 2, 2, -1, 2, -1, 2, -1, -1, -1, 2 ],
  [ 0, 0, 2, 0, 0, 0, 0, 0, 0, 2, 0, 0, -1, 2, 2, 2, -1, -1, -1, 2, 2, -1 ],
  [ 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 2, 0, 2, -1, -1, -1, 2, 2, -1, 2, 2, -1 ],
  [ 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 2, -1, 2, -1, -1, 2, -1, 2, 2, -1, 2 ],
  [ 0, 0, 0, 0, 0, 0, 0, 2, 2, 0, 0, 0, -1, -1, 2, -1, -1, 2, 2, -1, 2, 2 ],
  [ 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0 ],
  [ 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1 ],
  [ 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1 ],
  [ 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1 ],
  [ 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0 ],
  [ 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0 ],
  [ 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0 ],
  [ 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0 ],
  [ 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 1 ],
  [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1 ],
  [ 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0 ],
  [ 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0 ],
  [ 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0 ],
  [ 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1 ] ]

```

```

gap> Determinant(p);
1
gap> StablyPermutationFCheckMat(G,Nlist(1),Plist(1),p);
true

gap> U432[7]; # checking [M]^{f1}=0
[ 4, 31, 6, 2 ]
gap> G:=MatGroupZClass(4,31,6,2);; # G=C2xA5
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 16
16

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),5);;
gap> Set(List(mis,Length))-4; # Method III could not apply
[ 16, 26, 36, 46, 56, 66, 76, 86, 96, 106, 116, 126, 136, 146, 156, 166 ]

gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 0, 0, -2, 0, 0, 0, 0, -1, 0, -1, 1, 0, 0, 1, 0, -1, -1, 1, 2, 1, -2, 1 ],
  [ 0, 1, 0, 0, 0, 0, 0, 0, -2, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, -1, 0 ],
  [ 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, 1, 1, 1, -1, 0 ],
  [ 0, 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 0, 1, -1, 0, 0, 0, 0, 1 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, -1, 0, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, -1, 0, 0, 0, -1 ]
gap> bp:=StablyPermutationFCheckP(G,Nlist(1),Plist(1));;
gap> Length(bp);
11
gap> Length(bp[1]); # rank of the both sides of (10) is 22
22
gap> rs:=RandomSource(IsMersenneTwister);
<RandomSource in IsMersenneTwister>
gap> rr:=List([1..100000],x->List([1..11],y->Random(rs,[-1..2]))));;
gap> Filtered(rr,x->Determinant(x*bp)^2=1);
[ [ 2, 1, 0, 1, 0, 0, -1, 0, 0, 0, -1 ], [ 2, 0, 1, 0, 1, -1, 0, 1, 0, 1, 1 ],
  [ 2, 0, 1, 0, 1, -1, 0, 1, 0, 1, 1 ], [ 2, 0, 0, 1, 1, -1, -1, 0, 0, -1, -1 ],
  [ 2, 1, 2, -1, 0, 0, -1, 0, -1, 1, 2 ], [ 2, 1, -1, 0, 0, 0, 0, -1, 0, 0, 1 ] ]
gap> p:=last[1]*bp;
[ [ 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1 ],
  [ 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0 ],
  [ 1, 1, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1 ],
  [ 1, 1, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0 ],
  [ 1, 1, 1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1 ],
  [ 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0 ],
  [ 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0 ],
  [ -1, 0, 0, 0, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1 ],
  [ -1, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0 ],
  [ 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0 ],
  [ 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0 ],
  [ 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, -1, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, -1, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1 ],
  [ 0, -1, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0 ],
  [ 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 0, 0, -1, 0, 0, 0, -1, 0, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0 ],
  [ -1, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, -1, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0 ] ]

```

```

[ -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, -1, 0 ],
[ 0, -1, 0, 0, 0, -1, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0 ],
[ 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0 ] ]
gap> Determinant(p);
-1
gap> StablyPermutationFCheckMat(G,Nlist(1),Plist(1),p);
true

gap> i432:=Difference(U432,[U432[4],U432[7]]);
[ [ 4, 31, 1, 3 ], [ 4, 31, 1, 4 ], [ 4, 31, 2, 2 ], [ 4, 31, 4, 2 ],
  [ 4, 31, 5, 2 ], [ 4, 31, 7, 2 ] ]

gap> I4:=Union(i411,i432);
[ [ 4, 31, 1, 3 ], [ 4, 31, 1, 4 ], [ 4, 31, 2, 2 ], [ 4, 31, 4, 2 ],
  [ 4, 31, 5, 2 ], [ 4, 31, 7, 2 ], [ 4, 33, 2, 1 ] ]
gap> List(I4,x->StructureDescription(MatGroupZClass(x[1],x[2],x[3],x[4])));
[ "C5 : C4", "C5 : C4", "C2 x (C5 : C4)", "S5", "S5", "C2 x S5", "C3 : C8" ]
gap> Length(I4);
7

```

**Example 9.4** ( $[M_G]^{fl} = 0$  if and only if  $[M_G]^{fl}$  is of finite order in  $C(G)/S(G)$  for the groups  $G$  of the GAP codes (4,31,1,3), (4,31,1,4) and (4,33,2,1)).

```

gap> Read("Hn.gap");

gap> G:=MatGroupZClass(4,31,1,3);; # G=F20
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 16
16
gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 1, 0, 1, -1, 0, -2 ] ]
gap> l:=ll[1];
[ 1, 1, 0, 1, -1, 0, -2 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C4", "C5", "D10", "C5 : C4" ]
gap> gg:=StablyPermutationFCheckGen(G,Nlist(1),Plist(1));;
gap> G1:=Group(gg[1]);;
gap> G2:=Group(gg[2]);;
gap> [Length(G1.1),Length(G2.1)]; # rank of the both sides is 34
34
gap> TateCohomology(G1,2);
[ 2 ]
gap> TateCohomology(G2,2);
[ 10 ]

gap> G:=MatGroupZClass(4,31,1,4);; # G=F20
gap> ll:=PossibilityOfStablyPermutationF(G); # F is of rank 16
[ [ 1, 1, 0, 1, -1, 0, -2 ] ]
gap> l:=ll[1];
[ 1, 1, 0, 1, -1, 0, -2 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C4", "C5", "D10", "C5 : C4" ]
gap> gg:=StablyPermutationFCheckGen(G,Nlist(1),Plist(1));;
gap> G1:=Group(gg[1]);;
gap> G2:=Group(gg[2]);;

```

```

gap> [Length(G1.1),Length(G2.1)]; # rank of the both sides is 34
34
gap> TateCohomology(G1,2);
[ 5, 10 ]
gap> TateCohomology(G2,2);
[ 10 ]

gap> G:=MatGroupZClass(4,33,2,1);; # G=C3:C8
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 20
20
gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 1, 1, 0, -1, 0, 0, 0, -2 ] ]
gap> l:=ll[1];
[ 1, 1, 1, 0, -1, 0, 0, 0, -2 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C3", "C4", "C6", "C8", "C12", "C3 : C8" ]
gap> gg:=StablyPermutationFCheckGen(G,Nlist(l),Plist(l));;

gap> G1:=Group(gg[1]);
<matrix group with 2 generators>
gap> G2:=Group(gg[2]);
<matrix group with 2 generators>
gap> [Length(G1.1),Length(G2.1)]; # rank of the both sides is 44
[ 44, 44 ]
gap> List([-7..7],i->TateCohomology(G1,i)); # Tate cohomologies are coincide
[ [ ], [ 6 ], [ ], [ 6 ], [ ], [ 6 ], [ 0, 0, 0 ], [ 1, 1, 6 ],
  [ ], [ 6 ], [ ], [ 6 ], [ ], [ 6 ], [ ] ]
gap> List([-7..7],i->TateCohomology(G2,i));
[ [ ], [ 6 ], [ ], [ 6 ], [ ], [ 6 ], [ 0, 0, 0 ], [ 1, 1, 6 ],
  [ ], [ 6 ], [ ], [ 6 ], [ ], [ 6 ], [ ] ]

gap> Length(GeneratorsOfGroup(G)); # number of generators of G is 2
2
gap> List([G.1,G.2],Order); # order of two generators are 8, 12
[ 8, 12 ]
gap> a:=G.2^4;
[ [ -1, -1, -1, 1 ], [ 1, 0, 0, -1 ], [ 0, 0, -1, 1 ], [ 0, 0, -1, 0 ] ]
gap> b:=G.1;
[ [ 0, 0, -1, 0 ], [ -1, 0, 0, 0 ], [ 1, 1, 1, -2 ], [ 0, 1, 0, -1 ] ]
gap> List([a,b],Order);
[ 3, 8 ]
gap> a^b=a^2;
true
gap> G=Group(a,b);
true

gap> G1_3:=Group(gg[1]*Z(3));;
gap> G2_3:=Group(gg[2]*Z(3));;
gap> v1:=VectorSpace(GF(3^2),(G1_3.2^4-IdentityMat(44)*Z(3)^0)^2);
<vector space over GF(3^2), with 44 generators>
gap> v2:=VectorSpace(GF(3^2),(G2_3.2^4-IdentityMat(44)*Z(3)^0)^2);
<vector space over GF(3^2), with 44 generators>
gap> List([v1,v2],Dimension);
[ 12, 12 ]

```

```

gap> Eigenvalues(GF(3^2),G1_3.1);
[ Z(3), Z(3)^0, Z(3^2)^5, Z(3^2)^6, Z(3^2)^7, Z(3^2), Z(3^2)^2, Z(3^2)^3 ]
gap> Eigenvalues(GF(3^2),G2_3.1);
[ Z(3), Z(3)^0, Z(3^2)^5, Z(3^2)^6, Z(3^2)^7, Z(3^2), Z(3^2)^2, Z(3^2)^3 ]
gap> e1:=Eigenspaces(GF(3^2),G1_3.1);
[ <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 4 generators>,
  <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 4 generators>,
  <vector space over GF(3^2), with 4 generators>,
  <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 4 generators> ]
gap> e2:=Eigenspaces(GF(3^2),G2_3.1);
[ <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 4 generators>,
  <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 4 generators>,
  <vector space over GF(3^2), with 4 generators>,
  <vector space over GF(3^2), with 7 generators>,
  <vector space over GF(3^2), with 4 generators> ]
gap> List(e1,x->Dimension(Intersection(x,v1)));
[ 2, 2, 2, 2, 2, 0, 2, 0 ]
gap> List(e2,x->Dimension(Intersection(x,v2)));
[ 2, 2, 1, 2, 1, 1, 2, 1 ]

```

**Example 9.5** ( $[M_G]^{f_l} + [J_{S_5/S_4}]^{f_l} = 0$  for the group  $G \simeq S_5$  of the GAP code (4,31,5,2)).

```

gap> ip:=InverseProjection([[4,31,4,2],[4,31,5,2]]);
[ <matrix group of size 120 with 3 generators>,
  <matrix group of size 14400 with 6 generators>,
  <matrix group of size 7200 with 5 generators> ]
gap> G:=ip[1]; # G=S5
<matrix group of size 120 with 3 generators>
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 32
32
gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 0, 0, 0, 0, -4, 0, -1, 1, 0, -3, 0, 0, -1, 0, 4, 4, 1, -4, 1 ],
  [ 0, 1, 0, 0, 0, -1, -1, 0, 0, 0, -1, 0, 0, 0, 1, 1, 1, 0, -1, 0 ],
  [ 0, 0, 1, 0, 0, -2, 0, 0, 1, 0, -1, 0, -1, -1, 0, 2, 2, 1, -2, 0 ],
  [ 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 1 ],
  [ 0, 0, 0, 0, 1, -2, 2, 0, 2, 1, -2, -2, -1, -2, -2, 2, 4, 1, -2, 0 ] ]
gap> l:=ll[4];
[ 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 1 ]
gap> Length(l);
20
gap> [l[4],l[13],l[20],l[8],l[9],l[10],l[11]];
[ 1, 1, 1, -1, -1, -1, -1 ]
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C4", "C5", "C6", "S3",
  "S3", "D8", "D10", "A4", "D12", "C5 : C4", "S4", "A5", "S5" ]
gap> [ss[4],ss[13],ss[8],ss[9],ss[10],ss[11]];
[ "C3", "D10", "C5", "C6", "S3", "S3" ]

```

```

gap> l2:=IdentityMat(Length(l))[Length(l)-1];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ]
gap> bp:=StablyPermutationFCheckP(G,Nlist(l)+l2,Plist(l)+l2);;
gap> Length(bp);
85
gap> Length(bp[1]); # rank of the both sides is 85
85

# after some efforts we may get
gap> n:=[ -1, 1, 1, 1, -1, -1, -1, 0, 0, 1, 1, 1, 0, 0, -1, 0, 0, 0, -1, 0,
> 0, -1, -1, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, -1, 0, 1, 0,
> -1, -1, 1, 0, 0, -1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0,
> 0, 0, 1, -1, -1, 0, -1, -1, -1, 1, 0, 1, -1, 1, -1, -1, 1, 1, 1, 1,
> 5, 2, -3, 3, -2 ]
gap> p:=n*bp;;
gap> StablyPermutationFCheckMat(G,Nlist(l)+l2,Plist(l)+l2,p);
true

```

**Example 9.6** ( $[M_G]^{fl} + [J_{F_{20}/C_4}]^{fl} = 0$  for the group  $G \simeq F_{20}$  of the GAP code (4, 31, 1, 4)).

```

gap> ip:=InverseProjection([[4,31,1,3],[4,31,1,4]]);
[ <matrix group of size 20 with 3 generators>,
  <matrix group of size 400 with 4 generators>,
  <matrix group of size 200 with 5 generators>,
  <matrix group of size 100 with 4 generators>,
  <matrix group of size 100 with 4 generators> ]
gap> G:=ip[1];; # G=F20
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 32
32
gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 1, 0, 1, -1, 0, -1 ] ]
gap> l:=ll[1];
[ 1, 1, 0, 1, -1, 0, -1 ]
gap> List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C4", "C5", "D10", "C5 : C4" ]
gap> bp:=StablyPermutationFCheckP(G,Nlist(l),Plist(l));;
gap> Length(bp);
62
gap> Length(bp[1]); # rank of the both sides of (10) is 34
34

# after some efforts we may get
gap> n:=[ 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, -1, 0, 1, 0, -1, 1, 1,
> 0, 0, -1, -1, 0, 1, 1, -1, 0, 1, -1, -1, 0, 0, 1, -1, 0, 1, 1, 0,
> 0, 0, 0, 1, 1, -1, 0, -1, 1, -1, 1, 0, 1, 1, 0, -1, -1, 1, -1, -1,
> 0, -1 ];
gap> p:=n*bp;;
gap> Determinant(p);
1
gap> StablyPermutationFCheckMat(G,Nlist(l),Plist(l),p);
true

```

## 10. PROOF OF THEOREM 1.26

Let  $G$  be a finite subgroup of  $\mathrm{GL}(5, \mathbb{Z})$  and  $M = M_G$  be the corresponding  $G$ -lattice of rank 5 as in Definition 1.24.

**Step 1.** The case where  $M_G$  is decomposable.

We assume that  $M_G \simeq M_1 \oplus M_2$  is decomposable with  $\mathrm{rank} M_1 \geq \mathrm{rank} M_2 \geq 1$ . Hence  $\mathrm{rank} M_1 \leq 4$  and  $\mathrm{rank} M_2 \leq 2$ . It follows from Theorem 1.1 and Lemma 2.14 that  $[M_2]^{fl} = 0$  and  $[M_G]^{fl} = [M_1]^{fl}$ .

There exist exactly 25  $G$ -lattices  $M_G \simeq M_1 \oplus M_2$  with  $[M_G]^{fl} \neq 0$  but invertible whose GAP codes  $\mathcal{I}_{41}$  are computed in Example 4.12 (see also Table 11).

There exist exactly 245 (resp. 849, 768)  $G$ -lattices  $M_G \simeq M_1 \oplus M_2 \oplus M_3$  with  $\mathrm{rank} M_1 = 3$ ,  $\mathrm{rank} M_2 = 1$ ,  $\mathrm{rank} M_3 = 1$  (resp.  $M_G \simeq M_1 \oplus M_2$  with  $\mathrm{rank} M_1 = 3$ ,  $\mathrm{rank} M_2 = 2$ ,  $M_G \simeq M_1 \oplus M_2$  with  $\mathrm{rank} M_1 = 4$ ,  $\mathrm{rank} M_2 = 1$ ) and  $[M_G]^{fl} \neq 0$  whose GAP codes  $\mathcal{N}_{311}$  (resp.  $\mathcal{N}_{32}$ ,  $\mathcal{N}_{41}$ ) are computed in Example 4.12 (see also Tables 12, 13 and 14).

**Step 2.** Determination of all  $G$ -lattices  $M_G$  whose flabby class  $[M_G]^{fl}$  is not invertible.

We assume that  $M_G$  is indecomposable. There exist 1452 indecomposable  $G$ -lattices  $M_G$  of rank 5 (see Example 4.9). By using `IsInvertibleF` as in Algorithm F2, we see that there exist exactly 1141  $G$ -lattices  $M_G$  with  $[M_G]^{fl}$  not invertible (see Example 10.1 below). In the next step, we will show that all the remaining 311 cases satisfy  $[M_G]^{fl} = 0$ .

**Step 3.** Verification of  $[M_G]^{fl} = 0$  for all the remaining 311 cases.

First, we see that for all the remaining 311  $G$ -lattices  $M_G$   $G$  is a subgroup of at least one of the 18 groups of the CARAT codes as in Table 9 (see Example 10.2 below). Hence by Lemma 2.17 it is enough to show that  $[M_G]^{fl} = 0$  for the 18 groups  $G$  in Table 9.

Table 9: the maximal 18 groups in the remaining 311 cases

CARAT code	$G$	# $G$	Method	CARAT code	$G$	# $G$	Method
(5, 942, 1)	$\mathrm{Imf}(5, 1, 1)$	3840	<code>f1f1</code>	(5, 947, 2)	$S_5$	120	Method III + $\alpha$
(5, 953, 4)	$S_6$	720	<code>f1f1</code>	(5, 337, 12)	$D_4 \times S_3$	48	Method III
(5, 726, 4)		384	<code>f1f1</code>	(5, 341, 6)	$D_4 \times S_3$	48	Method III
(5, 919, 4)	$C_2 \times S_5$	240	Method III	(5, 531, 13)	$C_2 \times S_4$	48	Method III
(5, 801, 3)	$C_2 \times (S_3^2 \rtimes C_2)$	144	Method III	(5, 533, 8)	$C_2 \times S_4$	48	Method III
(5, 655, 4)	$D_4^2 \rtimes C_2$	128	<code>f1f1</code>	(5, 623, 4)	$C_2 \times S_4$	48	Method III
(5, 911, 4)	$S_5$	120	Method I (1)	(5, 245, 12)	$C_2^2 \times S_3$	24	Method III
(5, 946, 2)	$S_5$	120	Method I (2)	(5, 81, 42)	$C_2 \times D_4$	16	<code>f1f1</code>
(5, 946, 4)	$S_5$	120	Method II + $\alpha$	(5, 81, 48)	$C_2 \times D_4$	16	<code>f1f1</code>

Using `f1f1` as in Algorithm F3 once or twice, we see that  $[M_G]^{fl} = 0$  for the 6 groups  $G$  of the CARAT codes (5, 942, 1), (5, 953, 4), (5, 726, 4), (5, 655, 4), (5, 81, 42) and (5, 81, 48) (see Example 10.3 below).

Using functions in Algorithm F7: Method III, we may confirm that  $[M_G]^{fl} = 0$  for the 8 groups  $G$  of the CARAT codes (5, 919, 4), (5, 801, 3), (5, 337, 12), (5, 341, 6), (5, 531, 13), (5, 533, 8), (5, 623, 4) and (5, 245, 12) (see Example 10.4 below). Thus there are 4 remaining cases which are groups  $G \simeq S_5$  of order 120 of the CARAT codes (5, 911, 4), (5, 946, 2), (5, 946, 4), (5, 947, 2).

For two group  $G$  of the CARAT codes (5, 911, 4) and (5, 946, 2), by Method I (1) and Method I (2) respectively, we have that  $[M_G]^{fl} = 0$  (see Example 10.5 below and Example 5.6). Indeed, for the group  $G$  of the CARAT code (5, 911, 4), we see that  $M_G$  is flabby and coflabby and hence stably permutation by Theorem 6.2. This implies  $[M_G]^{fl} = 0$ . We also see that for the group  $G$  of the CARAT code (5, 946, 2), the carat code of  $F$  with  $[F] = [M_G]^{fl}$  is (5, 911, 4). Hence we confirm that  $[M_G]^{fl} = 0$ .

For the group  $G \simeq S_5$  of the CARAT code (5, 947, 2), using Algorithm F7: Method III, we may find the flabby class  $[M_G]^{fl} = [F]$  where  $F$  is of rank 21. However, the rank of the both sides of (10) becomes 81. Thus we take another  $F$  of rank 25 (see Example 10.6 below). Then the rank of the both sides of (10) becomes 55. By using



**PossibilityOfStablyPermutationFFromBase**, it is possible that

$$(17) \quad \mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_{11}] \oplus \mathbb{Z}[G/H_{17}] \simeq \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{15}] \oplus F$$

where  $H_6 \simeq C_2^2$ ,  $H_{11} \simeq S_3$ ,  $H_{17} \simeq S_4$ ,  $H_{10} \simeq S_3$  and  $H_{15} \simeq D_6$  (the rank of the both sides is  $30 + 20 + 5 = 20 + 10 + 25 = 55$ ). We could not establish the isomorphism (17). However, as in Example 5.6 (Algorithm F5: Method I (2)), after adding  $\mathbb{Z}$  to the both sides of (17), we may confirm that the isomorphism

$$\mathbb{Z}[G/H_6] \oplus \mathbb{Z}[G/H_{11}] \oplus \mathbb{Z}[G/H_{17}] \oplus \mathbb{Z} \simeq \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{15}] \oplus F \oplus \mathbb{Z}$$

holds (see Example 10.6).

For the group  $G \simeq S_5$  of the CARAT code (5,946,4), Method III does not work well. We may obtain  $[M_G]^{fl} = [F]$  with  $F$  rank 17 by **FlabbyResolution(G).actionF**. Using **PossibilityOfStablyPermutationF**, it turns out that the isomorphism

$$(18) \quad \begin{aligned} &\mathbb{Z}[G/H_6] \oplus \mathbb{Z}[H_9] \oplus \mathbb{Z}[G/H_{11}] \oplus \mathbb{Z}[G/H_{16}] \oplus 2\mathbb{Z}[G/H_{17}] \oplus \mathbb{Z}[G/H_{18}] \\ &\simeq \mathbb{Z}[G/H_7] \oplus \mathbb{Z}[G/H_{10}] \oplus \mathbb{Z}[G/H_{14}] \oplus \mathbb{Z} \oplus F \end{aligned}$$

may occur where  $H_6 \simeq C_2^2$ ,  $H_9 \simeq H_{11} \simeq S_3$ ,  $H_{16} \simeq F_{20}$ ,  $H_{17} \simeq S_4$ ,  $H_{18} \simeq A_5$ ,  $H_7 \simeq C_4$ ,  $H_{10} \simeq C_6$ ,  $H_{14} \simeq A_4$  and  $H_{15} \simeq D_6$  (the rank of the both sides is  $30 + 20 + 20 + 6 + 2 \times 5 + 2 = 30 + 20 + 10 + 10 + 1 + 17 = 88$ ). After some efforts, we see that the isomorphism (18) actually holds (see Example 10.7 below).

**Step 4.**  $[M_G]^{fl} = 0$  if and only if  $[M_G]^{fl}$  is of finite order in  $C(G)/S(G)$ .

We should show that if  $G$  is one of the 25 groups as in Table 11, i.e. the 25 cases where  $[M_G]^{fl}$  is not zero but invertible, then  $[(M_G)^{\oplus r}]^{fl} \neq 0$  for any  $r \geq 1$  (see also Step 5 of Section 9). By Remark 1.12,  $[M_G]^{fl} = [M_1]^{fl}$  where  $M_G \simeq M_1 \oplus M_2$ ,  $M_1$  is a  $G/N_1$ -lattice of rank 4 which is one of the 7 cases as in Table 2. Note that  $[M_1]^{fl} = \rho_G(M_1) = 0$  if and only if  $\rho_{G/N_1}(M_1) = 0$  by Lemma 2.14. Hence the assertion follows from Step 5 of Section 9.  $\square$

**Example 10.1** (Determination of all the cases where  $[M_G]^{fl}$  is not invertible).

```
gap> Read("caratnumber.gap");
gap> Read("FlabbyResolution.gap");
gap> Read("KS.gap");

gap> ind5:=LatticeDecompositions(5:Carat)[NrPartitions(5)];;
gap> Length(ind5);
1452
gap> N5:=Filtered(ind5,x->IsInvertibleF(CaratMatGroupZClass(x[1],x[2],x[3]))=false);;
gap> Length(N5);
1141
```

**Example 10.2** (The maximal 18 groups in the remaining 311 cases).

```
gap> U5:=Difference(ind5,N5);;
gap> Length(U5);
311
gap> gg18:=[[5,942,1],[5,953,4],[5,726,4],[5,919,4],[5,801,3],[5,655,4],
> [5,911,4],[5,946,2],[5,946,4],[5,947,2],[5,337,12],[5,341,6],
> [5,531,13],[5,533,8],[5,623,4],[5,245,12],[5,81,42],[5,81,48]];;
gap> Length(gg18);
18
gap> gg18sub:=Union(Set(gg18,y->Set(ConjugacyClassesSubgroups2(y),
> x->CaratZClass(Representative(x)))));
gap> Difference(e5a,gg18sub);
[ ]
```



```

[ 0, 0, 0, 1, 0 ], [ 0, 0, 1, 0, 0 ], [ 0, 1, 0, 0, 0 ], [ 1, 0, 0, 0, 0 ],
[ 1, 1, -1, -1, 1 ], [ 1, 1, 0, -1, 0 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);;
gap> Length(ll);
32
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0,
  0, 0, 0, 0, 1, -1 ]
gap> bp:=StablyPermutationFCheckPFromBase(G,mi,Nlist(l),Plist(l));;
gap> Length(bp);
16
gap> Length(bp[1]); # rank of the both sides of (10) is 11
11
gap> rr:=Filtered(Tuples([0,1],16),x->DeterminantMat(x*bp)^2=1);;
gap> Length(rr);
104
gap> rr[1];
[ 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0 ]
gap> p:=rr[1]*bp;
[ [ 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0 ],
  [ 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1 ],
  [ 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0 ],
  [ 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0 ],
  [ 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0 ],
  [ 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0 ],
  [ 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0 ],
  [ 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0 ],
  [ 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],
  [ 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0 ] ]
gap> StablyPermutationFCheckMatFromBase(G,mi,Nlist(l),Plist(l),p);
true

gap> G:=CaratMatGroupZClass(5,337,12);; # G=D4xS3
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 17
17
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 34, 34, 22, 22, 22, 22, 22, 22, 22, 22, 22, 14, 14, 22, 22, 22, 22, 22, 22, 14,
  14, 22, 22, 22, 22, 22, 22, 16, 16, 16, 16, 16, 16, 12, 10, 12, 10 ]
gap> mi:=mis[Length(mis)]; # (new) F is of rank 5 (=10-5)
[ [ -1, 1, 1, 1, -1 ], [ -1, 1, 1, 2, -1 ], [ -1, 1, 2, 1, -1 ], [ 0, -1, -1, -1, 1 ],
  [ 0, 0, -1, -1, 2 ], [ 0, 0, 1, 1, -2 ], [ 0, 0, 1, 1, -1 ], [ 1, -1, -2, -1, 1 ],
  [ 1, -1, -1, -2, 1 ], [ 1, 0, -1, -1, 1 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);
[ [ 1, -1, -1, -1, -1, 0, 1, -1, 1, -2, 2 ], [ 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, -1 ]
gap> StablyPermutationFCheckFromBase(G,mi,Nlist(l),Plist(l));
[ [ 1, 1, 1, 0, 2 ],
  [ -1, 0, 0, 0, -1 ],
  [ 0, -1, 0, 0, -1 ],
  [ 0, 0, 0, 1, -1 ],

```

```

[ 0, 0, -1, -1, 0 ] ]

gap> G:=CeratMatGroupZClass(5,341,6);; # G=D4xS3
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 14
14
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 19, 19, 19, 19, 11, 19, 19, 11, 19, 19, 19, 19, 10, 10, 10, 11, 7, 8, 8 ]
gap> mi:=mis[Length(mis)-2]; # (new) F is of rank 2 (=7-5)
[ [ -1, 0, 1, 1, -1 ], [ -1, 1, 1, 1, -1 ], [ -1, 1, 1, 1, 0 ], [ 0, 0, 0, 1, -1 ],
  [ 0, 0, 1, 0, -1 ], [ 0, 0, 1, 1, -1 ], [ 0, 1, 1, 1, -1 ] ]
gap> FlabbyResolutionFromBase(G,mi).actionF; # (new) F is permutation
Group([ [ [ 1, 0 ], [ 0, 1 ] ], [ [ 1, 0 ], [ 0, 1 ] ], [ [ 0, 1 ], [ 1, 0 ] ] ])

gap> G:=CeratMatGroupZClass(5,531,13);; # G=C2xS4
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 15
15
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 18, 18, 14, 14, 14, 14, 12, 12, 10, 12, 6, 7, 12, 6, 7 ]
gap> mi:=mis[Length(mis)-1]; # (new) F is trivial of rank 1 (=6-5)
[ [ 0, 0, 0, 1, 0 ], [ 0, 0, 1, 1, 1 ], [ 0, 1, 0, 1, 1 ],
  [ 1, -1, -1, 0, -1 ], [ 1, 0, 0, 0, -1 ], [ 1, 0, 0, 0, 0 ] ]
gap> FlabbyResolutionFromBase(G,mi).actionF; # (new) F is trivial of rank 1
Group([ [ [ 1 ] ], [ [ 1 ] ] ])

gap> G:=CeratMatGroupZClass(5,533,8);; # G=C2xS4
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 44
44
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 29, 29, 15, 15, 15, 15 ]
gap> mi:=mis[Length(mis)]; # (new) F is of rank 10 (=15-5)
[ [ -1, 0, -1, 0, 1 ], [ -1, 0, -1, 1, 0 ], [ 0, 0, -1, 0, 1 ], [ 0, 0, -1, 1, 0 ],
  [ 0, 0, 0, 0, 1 ], [ 0, 0, 0, 1, 0 ], [ 0, 1, -1, 0, 1 ], [ 0, 1, -1, 0, 2 ],
  [ 0, 1, -1, 1, 0 ], [ 0, 1, -1, 1, 1 ], [ 0, 1, -1, 2, 0 ], [ 0, 1, 0, 0, 1 ],
  [ 0, 1, 0, 1, 0 ], [ 1, 1, 0, 0, 1 ], [ 1, 1, 0, 1, 0 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);;
gap> Length(ll);
5
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0,
  0, -1, 0, 0, 0, 1, -1 ]
gap> bp:=StablyPermutationFCheckPFromBase(G,mi,Nlist(l),Plist(l));;
gap> Length(bp);
20
gap> Length(bp[1]); # rank of the both sides of (10) is 13
13
gap> rr:=Filtered(Tuples([0,1],20),x->DeterminantMat(x*bp)^2=1);;
gap> Length(rr);
2448
gap> rr[1];
[ 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1 ]
gap> p:=rr[1]*bp;
[ [ 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],

```

```

[ 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1 ],
[ 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1 ],
[ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0 ],
[ 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 ],
[ 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0 ],
[ 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0 ],
[ 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],
[ 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0 ],
[ 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],
[ -2, -2, -2, -2, -2, -2, -1, -1, -1, -1, -1, -1, -2 ],
[ 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0 ] ]
gap> StablyPermutationFCheckMatFromBase(G,mi,Nlist(1),Plist(1),p);
true

gap> G:=CaratMatGroupZClass(5,623,4);; # G=C2xS4
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 13
13
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 42, 30, 24, 33, 21, 15, 36, 38, 42, 33, 42, 36, 30, 24, 18, 26, 20, 30, 24, 21,
  15, 24, 18, 12 ]
gap> mi:=mis[Length(mis)]; # (new) F is of rank 7 (=12-5)
[ [ -1, -1, 1, 0, 0 ], [ -1, 0, 1, 0, 0 ], [ -1, 1, -1, 1, -1 ], [ -1, 1, 0, 1, -1 ],
  [ 0, -1, 1, -1, 1 ], [ 0, 0, 1, -1, 1 ], [ 0, 1, -1, 1, -1 ], [ 0, 1, 0, 1, -1 ],
  [ 1, -1, 0, -1, 1 ], [ 1, -1, 0, 0, 1 ], [ 1, 0, -1, -1, 0 ], [ 1, 0, -1, 0, 0 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);
[ [ 0, 2, 1, -1, -1 ] ]
gap> l:=ll[1];
[ 0, 2, 1, -1, -1 ]
gap> bp:=StablyPermutationFCheckPFromBase(G,mi,Nlist(1),Plist(1));;
gap> Length(bp);
14
gap> Length(bp[1]); # rank of the both sides of (10) is 8
8
gap> rr:=Filtered(Tuples([0,1],14),x->DeterminantMat(x*bp)^2=1);;
gap> Length(rr);
224
gap> rr[1];
[ 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1 ]
gap> p:=rr[1]*bp;
[ [ 0, 0, 0, 1, 1, 1, 1, 1 ],
  [ 1, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 1, 0, 1, 0, 0, 1 ],
  [ 0, 1, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 1, 1, 0, 0, 1, 0 ],
  [ 0, 1, 0, 0, 0, 1, 1, 0 ],
  [ 1, 0, 0, 0, 0, 1, 0, 1 ],
  [ 0, 1, 0, 1, 0, 0, 0, 1 ] ]
gap> StablyPermutationFCheckMatFromBase(G,mi,Nlist(1),Plist(1),p);
true

gap> G:=CaratMatGroupZClass(5,245,12);; # G=C2^2xS3
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 16
16

```

```

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 11 ]
gap> mi:=mis[1]; # (new) F is of rank 6 (=11-5)
[ [ -1, -1, 1, -2, 0 ], [ -1, 0, 1, -2, -1 ], [ 0, 0, -1, 1, 1 ], [ 0, 0, 0, -1, -1 ],
  [ 0, 0, 0, 1, 1 ], [ 0, 0, 1, -2, -1 ], [ 0, 0, 1, -1, -1 ], [ 0, 1, -1, 2, 1 ],
  [ 0, 1, 0, 0, 0 ], [ 1, 1, -1, 2, 0 ], [ 1, 1, -1, 2, 1 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);;
gap> Length(ll);
8
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
  1, 0, 0, 1, 0, 0, -1, -1 ]
gap> StablyPermutationFCheckFromBase(G,mi,Nlist(l),Plist(l));
[ [ -1, -1, -1, 0, 0, -1, -1 ],
  [ 1, 0, 1, 0, 0, 0, 1 ],
  [ 1, 1, 1, 0, -1, 1, 0 ],
  [ 0, 1, 1, 0, 0, 1, 0 ],
  [ 1, 1, 0, 0, 0, 0, 1 ],
  [ 1, 1, 0, 0, 0, 1, 0 ],
  [ 1, 1, 1, -1, 0, 1, 0 ] ]

```

**Example 10.5** (Verification of  $[M_G]^{fl} = 0$  for two groups  $G \simeq S_5$  of the CARAT codes (5, 911, 4) and (5, 946, 2)).

```

gap> G:=CaratMatGroupZClass(5,911,4);; # G=S5
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 11
11
gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 36, 16, 16, 16, 16, 6 ]
gap> mi:=mis[Length(mis)]; # (new) F is of rank 1 (=6-5)
[ [ 0, 0, -1, 0, 1 ], [ 0, 0, 0, -1, 1 ], [ 0, 1, 0, -1, 0 ],
  [ 1, -1, -1, 0, 1 ], [ 1, 0, -1, -1, 2 ], [ 1, 0, 0, 0, 1 ] ]
gap> FlabbyResolutionFromBase(G,mi).actionF; # (new) F is trivial
Group([ [ [ 1 ] ], [ [ 1 ] ] ])

gap> G:=CaratMatGroupZClass(5,946,2);; # G=S5
gap> CaratZClass(FlabbyResolution(G).actionF);
[ 5, 911, 4 ]

```

**Example 10.6** (Verification of  $[M_G]^{fl} = 0$  for the group  $G \simeq S_5$  of the CARAT code (5, 947, 2)).

```

gap> G:=CaratMatGroupZClass(5,947,2);; # G=S5
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 45
45

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),3);; # Method III
gap> List(mis,Length);
[ 51, 81, 51, 51, 41, 105, 75, 75, 65, 45, 96, 66, 66, 56, 36, 96, 96, 86, 66, 66,
  56, 36, 56, 36, 26, 120, 120, 110, 90, 90, 80, 60, 80, 60, 50, 30 ]
gap> mi26:=mis[25];; # (new) F is of rank 21 (=26-5)
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi26);;
gap> Length(ll);
5

```

```

gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 1, -1, 0, 1, -1, 1, 0, 0, 0, -1, 1, 1, 0, 0, -1 ]
gap> bp:=StablyPermutationFCheckPFromBase(G,mi26,Nlist(l),Plist(l));;
gap> Length(bp[1]); # but the rank of the both sides of (10) is 81
81

gap> mi:=mis[Length(mis)]; # (new) F is of rank 25 (=30-5)
[ [ -2, -1, -1, 0, 1 ], [ -2, 0, -1, 1, 1 ], [ -1, -2, 0, -1, 1 ], [ -1, -1, -1, -1, 1 ],
  [ -1, -1, -1, 0, 0 ], [ -1, -1, 0, -1, 0 ], [ -1, 0, 0, 1, 1 ], [ -1, 1, -1, 1, 1 ],
  [ -1, 1, -1, 2, 0 ], [ -1, 1, 0, 1, 0 ], [ 0, -2, 1, -1, 1 ], [ 0, -1, 1, 0, 1 ],
  [ 0, 0, -1, -1, 1 ], [ 0, 0, -1, 1, -1 ], [ 0, 0, 1, -1, -1 ], [ 0, 1, -1, 0, 1 ],
  [ 0, 1, -1, 2, -1 ], [ 0, 1, 1, 0, -1 ], [ 1, -1, 1, -1, 1 ], [ 1, -1, 1, 0, 0 ],
  [ 1, -1, 2, -1, 0 ], [ 1, 0, 0, -1, 1 ], [ 1, 0, 0, 1, -1 ], [ 1, 0, 2, -1, -1 ],
  [ 1, 1, -1, 0, 0 ], [ 1, 1, -1, 1, -1 ], [ 1, 1, 0, -1, 0 ], [ 1, 1, 0, 1, -2 ],
  [ 1, 1, 1, -1, -1 ], [ 1, 1, 1, 0, -2 ] ]
gap> ll:=PossibilityOfStablyPermutationFFromBase(G,mi);
[ [ 1, 0, 0, -1, 0, 0, -4, 0, 1, -2, 2, 0, -1, -1, 0, 4, 4, 1, -4, 0 ],
  [ 0, 1, 0, 0, 0, 0, -1, 0, 0, -2, 1, 0, 0, 0, 0, 1, 2, 0, -1, -1 ],
  [ 0, 0, 1, 0, 0, 0, -2, 0, 0, -1, 1, 0, -1, -1, 0, 2, 2, 1, -2, 0 ],
  [ 0, 0, 0, 0, 1, 0, -2, 0, 1, 0, 0, -2, -1, -2, 0, 2, 2, 1, -2, 2 ],
  [ 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 1, 0, 0, 0, -1, 0, 1, 0, 0, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 1, 0, 0, 0, -1, 0, 1, 0, 0, -1 ]
gap> Length(l);
20
gap> [l[6],l[11],l[17],l[10],l[15],l[20]];
[ 1, 1, 1, -1, -1, -1 ]
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C4", "C5", "C6", "S3", "S3",
  "D8", "D10", "A4", "D12", "C5 : C4", "S4", "A5", "S5" ]
gap> Length(ss);
19
gap> [ss[6],ss[11],ss[17],ss[10],ss[15]];
[ "C2 x C2", "S3", "S4", "S3", "D12" ]
gap> bp:=StablyPermutationFCheckPFromBase(G,mi,Nlist(l),Plist(l));;
gap> Length(bp);
40
gap> Length(bp[1]); # rank of the both sides of (10) is 55
55

gap> l2:=IdentityMat(Length(l))[Length(l)-1];
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ]
gap> bp:=StablyPermutationFCheckPFromBase(G,mi,Nlist(l)+l2,Plist(l)+l2);;
gap> Length(bp);
47
gap> Length(bp[1]); # rank of the both sides of (10) is 56
56

# after some efforts we may get
gap> n:= [ 1, 0, -1, 0, -1, -1, -1, 1, 0, 0, -1, 1, -1, 0, -1, 1, 1, 1, 1, 0,
> 0, 1, 0, 1, 0, -1, -1, -1, 0, -1, 1, 2, 0, -1, 0, 0, 1, 1, 1, -1,
> -1, 0, -1, 0, 1, 0, 1 ];
gap> p:=n*bp;;
gap> Determinant(p);
-1

```

```
gap> StablyPermutationFCheckMatFromBase(G,mi,Nlist(1)+12,Plist(1)+12,p));
true
```

**Example 10.7** (Verification of  $[M_G]^{fl} = 0$  for the group  $G \simeq S_5$  of the CARAT code (5, 946, 4)).

```
gap> G:=CaratMatGroupZClass(5,946,4);; # G=S5
gap> Rank(FlabbyResolution(G).actionF.1); # F is of rank 17
17

gap> mis:=SearchCoflabbyResolutionBase(TransposedMatrixGroup(G),5);;
gap> Set(List(mis,Length))-5; # Method III could not apply
[ 17, 32, 35, 47, 50, 62, 65, 77, 80, 92, 95 ]

gap> ll:=PossibilityOfStablyPermutationF(G);
[ [ 1, 0, 0, -1, 0, 0, -4, 0, 1, -2, 2, 0, -1, -1, 0, 4, 4, 1, -4, 0 ],
  [ 0, 1, 0, 0, 0, 0, -2, 0, 1, -2, 1, 0, 0, -1, 0, 2, 3, 1, -2, -1 ],
  [ 0, 0, 1, 0, 0, 0, -2, 0, 0, -1, 1, 0, -1, -1, 0, 2, 2, 1, -2, 0 ],
  [ 0, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, -2, -1, 0, 0, 0, 0, -1, 0, 2 ],
  [ 0, 0, 0, 0, 0, 1, -1, 0, 1, -1, 1, 0, 0, -1, -1, 1, 2, 1, -1, -1 ] ]
gap> l:=ll[Length(ll)];
[ 0, 0, 0, 0, 0, 1, -1, 0, 1, -1, 1, 0, 0, -1, -1, 1, 2, 1, -1, -1 ]
gap> Length(l);
20
gap> [l[6],l[9],l[11],l[16],l[17],l[18],l[7],l[10],l[14],l[15],l[19],l[20]];
[ 1, 1, 1, 1, 2, 1, -1, -1, -1, -1, -1, -1 ]
gap> ss:=List(ConjugacyClassesSubgroups2(G),x->StructureDescription(Representative(x)));
[ "1", "C2", "C2", "C3", "C2 x C2", "C2 x C2", "C4", "C5", "S3", "C6", "S3",
  "D8", "D10", "A4", "D12", "C5 : C4", "S4", "A5", "S5" ]
gap> [ss[6],ss[9],ss[11],ss[16],ss[17],ss[18],ss[7],ss[10],ss[14],ss[15],ss[19]];
[ "C2 x C2", "S3", "S3", "C5 : C4", "S4", "A5", "C4", "C6", "A4", "D12", "S5" ]
gap> bp:=StablyPermutationFCheckP(G,Nlist(1),Plist(1));;
gap> Length(bp);
122
gap> Length(bp[1]); # rank of the both sides of (10) is 88
88

# after some efforts we may get
gap> n:=[ -1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, -1, 1, -1, 0, -1, 0, 0,
> -1, 1, 0, -1, 1, 1, 1, 0, 0, 0, -1, -1, 0, -1, 1, 0, 1, 1, -1, 0,
> 1, 1, 0, -1, 1, 0, 1, 1, -1, 0, 1, 1, -1, 0, -1, 1, -1, -1, 0, 1,
> 1, 0, 1, -1, 1, -1, -1, 0, 1, -1, 0, 0, 0, -1, 1, 0, -1, -1, -1, -1,
> 0, -1, -1, 1, 1, 1, 0, 2, -2, 4, 0, 1, 3, -1, -1, -1, -1, -1, 0, -1,
> -1, -1, -1, 1, 0, 1, -1, 0, -1, 1, 0, -1, -1, 1, -1, 0, 0, -1, 1, 1,
> -1, 1 ];;
gap> p:=n*bp;;
gap> Determinant(p);
-1
gap> StablyPermutationFCheckMat(G,Nlist(1),Plist(1),p);
true
```



## 11. PROOF OF THEOREM 11.3

**Theorem 11.1** (Yamasaki [Yam12, Lemma 4.3]). *Let  $k$  be a field of char  $k \neq 2$  and  $k(x, y, z)$  be the rational function field over  $k$  with variables  $x, y, z$ . Let  $\sigma$  be a  $k$ -involution on  $k(x, y, z)$  defined by*

$$\sigma : x \mapsto -x, \quad y \mapsto \frac{a}{y}, \quad z \mapsto \frac{-bx^2 + c}{z} \quad (a, b, c \in k^\times).$$

(i)  $k(x, y, z)^{\langle \sigma \rangle} = k(z_0, z_1, z_2, z_3)$  where

$$z_0^2 = (z_1^2 - a)(z_2^2 - b)(z_3^2 - c).$$

(ii) *The fixed field  $k(x, y, z)^{\langle \sigma \rangle}$  is  $k$ -rational if and only if  $[k(\sqrt{a}, \sqrt{b}, \sqrt{c}) : k] \leq 2$  or  $[k(\sqrt{a}, \sqrt{b}, \sqrt{c}) : k] = 4$  with  $abc \notin k^{\times 2}$ . In particular, if  $k(x, y, z)^{\langle \sigma \rangle}$  is not  $k$ -rational, then  $k(x, y, z)^{\langle \sigma \rangle}$  is not retract  $k$ -rational.*

**Example 11.2** (Another proof of Theorem 11.1: not retract  $k$ -rational cases). Assume that  $[k(\sqrt{a}, \sqrt{b}, \sqrt{c}) : k] = 4$  and  $abc \in k^{\times 2}$ . We will show that  $k(x, y, z)^{\langle \sigma \rangle}$  is not retract  $k$ -rational by using `IsInvertibleF` in Algorithm F2. This also implies that  $k(x, y, z)^{\langle \sigma \rangle}$  is not retract  $k$ -rational when  $[k(\sqrt{a}, \sqrt{b}, \sqrt{c}) : k] = 8$ .

We may assume that  $c = ab$ . Put  $\alpha = \sqrt{a}$ ,  $\beta = \sqrt{b}$  and  $L = k(\alpha, \beta)$ . Then  $L = k(\sqrt{a}, \sqrt{b}, \sqrt{c})$  and  $[L : k] = 4$ . Put  $y' := \frac{y-\alpha}{y+\alpha}$ ,  $z' := \frac{z-\beta x-\alpha\beta}{z+\beta x+\alpha\beta}$ . Then  $L(x, y, z) = L(x, y', z')$  and  $\sigma$  acts on  $L(x, y', z')$  by  $\sigma : x \mapsto -x$ ,  $y' \mapsto -y'$ ,  $z' \mapsto -z'$ . Put  $y_1 := x^2$ ,  $y_2 := xy'$ ,  $y_3 := xz'$ . Then  $k(x, y, z)^{\langle \sigma \rangle} = (L(x, y, z)^{\langle \sigma \rangle})^{\langle \rho_a, \rho_b \rangle} = L(y_1, y_2, y_3)^{\langle \rho_a, \rho_b \rangle}$  is  $L$ -rational where

$$\begin{aligned} \rho_a : \alpha &\mapsto -\alpha, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{y_1}{y_2}, \quad y_3 \mapsto \frac{y_1(y_3 + \alpha)}{y_1 + \alpha y_3}, \\ \rho_b : \beta &\mapsto -\beta, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto \frac{y_1}{y_3}. \end{aligned}$$

Let  $G = \text{Gal}(L/k) = \langle \rho_a, \rho_b \rangle \simeq C_2 \times C_2$ . We consider the  $G$ -lattice  $M = \langle y_1, y_2, y_3, t_1, t_2, t_3 \rangle$  of rank 6 where  $(t_1, t_2, t_3) = (y_1 - \alpha, y_1 + \alpha y_3, y_3 + \alpha)$ . The action of  $G$  on  $L(M)$  is given by

$$\begin{aligned} \rho_a : \alpha &\mapsto -\alpha, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{y_1}{y_2}, \quad y_3 \mapsto \frac{y_1 t_3}{t_2}, \quad t_1 \mapsto t_1, \quad t_2 \mapsto \frac{y_1 t_1}{t_2}, \quad t_3 \mapsto \frac{y_3 t_1}{t_2}, \\ \rho_b : \beta &\mapsto -\beta, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto \frac{y_1}{y_3}, \quad t_1 \mapsto t_1, \quad t_2 \mapsto \frac{y_1 t_3}{y_3}, \quad t_3 \mapsto \frac{t_2}{y_3}. \end{aligned}$$

The actions of  $\rho_a$  and  $\rho_b$  on  $M$  are represented as matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

By Algorithm F2, we see that  $[M]^{fl}$  is not invertible. Hence  $L(M)^G$  is not retract  $k$ -rational. It follows from [Yam12, Theorem 2.10] that  $k(x, y, z)^{\langle \sigma \rangle}$  is not retract  $k$ -rational (cf. [Yam12, Case 3 in the proof of Lemma 4.3], in particular, we do not need to enlarge  $M$  in order to vanish  $H^1$ ).

```
gap> Read("FlabbyResolution.gap");
```

```
gap> v1:=[[1,0,0,0,0,0],[1,-1,0,0,0,0],[1,0,0,0,-1,1],
> [0,0,0,1,0,0],[1,0,0,1,-1,0],[0,0,1,1,-1,0]];;
gap> v2:=[[1,0,0,0,0,0],[0,1,0,0,0,0],[1,0,-1,0,0,0],
> [0,0,0,1,0,0],[1,0,-1,0,0,1],[0,0,-1,0,1,0]];;
gap> IsInvertibleF(Group(v1,v2));
false
```

We generalize Theorem 11.1 as follows:

**Theorem 11.3.** *Let  $k$  be a field of char  $k \neq 2$  and  $k(x, y, z)$  be the rational function field over  $k$  with variables  $x, y, z$ . Let  $\sigma_{a,b,c,d}$  be a  $k$ -involution on  $k(x, y, z)$  defined by*

$$\sigma_{a,b,c,d} : x \mapsto -x, \quad y \mapsto \frac{-ax^2 + b}{y}, \quad z \mapsto \frac{-cx^2 + d}{z} \quad (a, b, c, d \in k^\times)$$

and  $m = [k(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) : k]$ .

(i)  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = k(t_1, t_2, t_3, t_4)$  where  $t_1, t_2, t_3, t_4$  satisfy the relation

$$(19) \quad (t_1^2 - a)(t_4^2 - d) = (t_2^2 - b)(t_3^2 - c).$$

(ii)  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -isomorphic to  $k(x, y, z)^{\langle \sigma_{\tau(a), \tau(b), \tau(c), \tau(d)} \rangle}$  for  $\tau \in D_4$  where  $D_4 = \langle (abdc), (ab)(cd) \rangle$  is the permutation group on the set  $\{a, b, c, d\}$  which is isomorphic to the dihedral group of order 8.

(iii) If one of the following conditions holds, then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract  $k$ -rational:

(C15)  $m = 4$ , (1)  $ab, acd \in k^{\times 2}$ ; (2)  $bd, abc \in k^{\times 2}$ ; (3)  $cd, abd \in k^{\times 2}$ ; (4)  $ac, bcd \in k^{\times 2}$ ;

(C16)  $m = 4$ , (1)  $ad, abc \in k^{\times 2}$ ; (2)  $bc, abd \in k^{\times 2}$ ;

(C18)  $m = 8$ , (1)  $ab \in k^{\times 2}$ ; (2)  $ac \in k^{\times 2}$ ; (3)  $bd \in k^{\times 2}$ ; (4)  $cd \in k^{\times 2}$ ;

(C19)  $m = 8$ , (1)  $ad \in k^{\times 2}$ ; (2)  $bc \in k^{\times 2}$ ;

(C20)  $m = 8$ , (1)  $abc \in k^{\times 2}$ ; (2)  $bcd \in k^{\times 2}$ ; (3)  $abd \in k^{\times 2}$ ; (4)  $acd \in k^{\times 2}$ ;

(C21)  $m = 8$ ,  $abcd \in k^{\times 2}$ ;

(C22)  $m = 16$ .

*Proof of Theorem 11.3.* We prove the assertion (i). Put

$$\begin{aligned} t_1 &:= \frac{1}{2x} \left( y - \frac{-ax^2 + b}{y} \right), \quad t_2 := \frac{1}{2} \left( y + \frac{-ax^2 + b}{y} \right), \\ t_3 &:= \frac{1}{2x} \left( z - \frac{-cx^2 + d}{z} \right), \quad t_4 := \frac{1}{2} \left( z + \frac{-cx^2 + d}{z} \right). \end{aligned}$$

Then we see that  $k(t_1, t_2, t_3, t_4) \subset k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$ . It follows from the equalities

$$y = t_2 + t_1 x, \quad z = t_4 + t_3 x, \quad x^2(t_3^2 - c) - (t_4^2 - d) = 0$$

that  $[k(x, y, z) : k(t_1, t_2, t_3, t_4)] \leq 2$ . Hence we get  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = k(t_1, t_2, t_3, t_4)$ . The relation  $(t_1^2 - a)(t_4^2 - d) = (t_2^2 - b)(t_3^2 - c)$  may be obtained by the direct calculation. The assertion (ii) follows from (i). We will prove the assertion (iii).

The case (C15):  $m = 4$ . By (ii), we should show only the case (1)  $ab, acd \in k^{\times 2}$ . Define  $Y := \frac{ay}{ax + \sqrt{ab}}$ . Then  $k(x, y, z) = k(x, Y, z)$  and  $\sigma_{a,b,c,d}$  acts on  $k(x, Y, z)$  by

$$\sigma_{a,b,c,d} : x \mapsto -x, \quad Y \mapsto \frac{a}{Y}, \quad z \mapsto \frac{-cx^2 + d}{z}.$$

By Theorem 11.1 (ii),  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract  $k$ -rational.

The case (C16):  $m = 4$ . By (ii), we should treat only the case (1)  $ad, abc \in k^{\times 2}$ . Let  $L = k(\alpha, \beta, \gamma, \delta)$  where  $\alpha^2 = a$ ,  $\beta^2 = b$ ,  $\gamma^2 = c$ ,  $\delta^2 = d$ . Then  $L = k(\alpha, \beta)$  and  $[L : k] = 4$ . Put  $y' := (\alpha x + \beta)/y$ ,  $z' := (\gamma x + \delta)/z$ . Then  $L(x, y, z) = L(x, y', z')$  and  $\sigma_{a,b,c,d}$  acts on  $L(x, y', z')$  by

$$\sigma_{a,b,c,d} : x \mapsto x, \quad y' \mapsto \frac{1}{y'}, \quad z' \mapsto \frac{1}{z'}.$$

We put

$$y_1 := x^2, \quad y_2 := x \frac{1 - y'}{1 + y'} = x \frac{y - \alpha x - \beta}{y + \alpha x + \beta}, \quad y_3 := x \frac{1 - z'}{1 + z'} = x \frac{z - \gamma x - \delta}{y + \gamma x + \delta}.$$

By the assumptions  $ad, abc \in k^{\times 2}$ , there exist  $e, f \in k^\times$  such that  $\gamma = \alpha\beta e$  and  $\delta = \alpha f$ . Then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = (L(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle})^{\langle \rho_a, \rho_b \rangle} = L(y_1, y_2, y_3)^{\langle \rho_a, \rho_b \rangle}$ , where

$$\begin{aligned} \rho_a : \alpha &\mapsto -\alpha, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{\alpha y_1 + \beta y_2}{\alpha y_2 + \beta}, \quad y_3 \mapsto \frac{y_1}{y_3}, \\ \rho_b : \beta &\mapsto -\beta, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{y_1(\alpha y_2 + \beta)}{\alpha y_1 + \beta y_2}, \quad y_3 \mapsto \frac{\beta e y_1 + f y_3}{\beta e y_3 + f}. \end{aligned}$$

Let  $G = \text{Gal}(L/k) = \langle \rho_a, \rho_b \rangle \simeq C_2 \times C_2$ . We consider the  $G$ -lattice  $M = \langle y_1, y_2, y_3, t_1, t_2, t_3, t_4, u_1, u_2 \rangle$  of rank 9 where

$$(t_1, t_2, t_3, t_4, u_1, u_2) = (\alpha y_1 + \beta y_2, \alpha y_2 + \beta, \beta e y_1 + f y_3, \beta e y_3 + f, \alpha y_1 - b, -\beta e^2 y_1 + f^2).$$

The action of  $G$  on  $L(M)$  is given by

$$\begin{aligned}\rho_a : \alpha &\mapsto -\alpha, \quad y_2 \mapsto \frac{t_1}{t_2}, \quad y_3 \mapsto \frac{y_1}{y_3}, \quad t_1 \mapsto -\frac{u_1 y_2}{t_2}, \quad t_2 \mapsto -\frac{u_1}{t_2}, \quad t_3 \mapsto \frac{y_1 t_4}{y_3}, \quad t_4 \mapsto \frac{t_3}{y_3}, \\ \rho_b : \beta &\mapsto -\beta, \quad y_2 \mapsto \frac{y_1 t_2}{t_1}, \quad y_3 \mapsto \frac{t_3}{t_4}, \quad t_1 \mapsto \frac{y_1 u_1}{t_1}, \quad t_2 \mapsto \frac{u_1 y_2}{t_1}, \quad t_3 \mapsto \frac{u_2 y_3}{t_4}, \quad t_4 \mapsto \frac{u_2}{t_4}\end{aligned}$$

where  $y_1, u_1, u_2$  are invariants under the action of  $G$ . The actions of  $\rho_a$  and  $\rho_b$  on  $M$  are represented as matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Algorithm F2, we see that  $[M]^{fl}$  is not invertible (see Example 11.4 below). Hence  $L(M)^G$  is not retract  $k$ -rational. It follows from [Yam12, Theorem 2.10] that  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract  $k$ -rational.

The cases (C18), (C19) and (C20). By the result of (C15) and (C16),  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract rational over some quadratic extension of  $k$ , hence not retract  $k$ -rational.

The case (C21):  $m = 8$ ,  $abcd \in k^{\times 2}$ . Let  $L = k(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ . We assume  $[L : k] = 8$  and  $abcd \in k^{\times 2}$ , and hence  $d = abce^2$  for some  $e \in k^{\times}$ . We put  $\alpha := \sqrt{a}$ ,  $\beta := \sqrt{b}$ ,  $\gamma := \sqrt{c}$ . Then  $L = k(\alpha, \beta, \gamma)$ . We first see that  $L(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $L$ -rational as follows. Put

$$y_1 := x^2, \quad y_2 := x \frac{y - \alpha x - \beta}{y + \alpha x + \beta}, \quad y_3 := x \frac{z - \gamma x - \alpha \beta \gamma e}{z + \gamma x + \alpha \beta \gamma e}.$$

Then  $L(x, y, z) = L(x, y_2, y_3)$  and the action of  $\sigma_{a,b,c,d}$  on  $L(x, y_2, y_3)$  is given by  $\sigma_{a,b,c,d} : x \mapsto -x, y_2 \mapsto y_2, y_3 \mapsto y_3$ . Hence the field  $L(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = L(y_1, y_2, y_3)$  is  $L$ -rational. Let  $G = \langle \rho_a, \rho_b, \rho_c \rangle \simeq C_2 \times C_2 \times C_2$  be the Galois group  $\text{Gal}(L/k)$  of  $L/k$  where

$$\begin{aligned}\rho_a : \alpha &\mapsto -\alpha, \quad \beta \mapsto \beta, \quad \gamma \mapsto \gamma, \\ \rho_b : \alpha &\mapsto \alpha, \quad \beta \mapsto -\beta, \quad \gamma \mapsto \gamma, \\ \rho_c : \alpha &\mapsto \alpha, \quad \beta \mapsto \beta, \quad \gamma \mapsto -\gamma.\end{aligned}$$

Then we get  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = (L(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle})^G = L(y_1, y_2, y_3)^G$ . The action of  $G$  on  $L(y_1, y_2, y_3)$  is given by

$$\begin{aligned}\rho_a : \alpha &\mapsto -\alpha, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{\alpha y_1 + \beta y_2}{\alpha y_2 + \beta}, \quad y_3 \mapsto \frac{y_1(y_3 + \alpha \beta e)}{y_1 + \alpha \beta e y_3}, \\ \rho_b : \beta &\mapsto -\beta, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{y_1(\alpha y_2 + \beta)}{\alpha y_1 + \beta y_2}, \quad y_3 \mapsto \frac{y_1(y_3 + \alpha \beta e)}{y_1 + \alpha \beta e y_3}, \\ \rho_c : \gamma &\mapsto -\gamma, \quad y_1 \mapsto y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto \frac{y_1}{y_3}.\end{aligned}$$

We consider the  $G$ -lattice  $M = \langle y_1, y_2, y_3, t_1, t_2, t_3, t_4, u_1, u_2 \rangle$  of rank 9 where

$$(t_1, t_2, t_3, t_4, u_1, u_2) = (\alpha y_1 + \beta y_2, \alpha y_2 + \beta, y_3 + \alpha \beta e, y_1 + \alpha \beta e y_3, b - \alpha y_1, y_1 - a b e^2).$$

The action of  $G$  on  $L(M)$  is given by

$$\begin{aligned}\rho_a : \alpha &\mapsto -\alpha, \quad y_2 \mapsto \frac{t_1}{t_2}, \quad y_3 \mapsto \frac{y_1 t_3}{t_4}, \quad t_1 \mapsto \frac{u_1 y_2}{t_2}, \quad t_2 \mapsto \frac{u_1}{t_2}, \quad t_3 \mapsto \frac{u_2 y_3}{t_4}, \quad t_4 \mapsto \frac{y_1 u_2}{t_4}, \\ \rho_b : \beta &\mapsto -\beta, \quad y_2 \mapsto \frac{y_1 t_2}{t_1}, \quad y_3 \mapsto \frac{y_1 t_3}{t_4}, \quad t_1 \mapsto -\frac{y_1 u_2}{t_1}, \quad t_2 \mapsto -\frac{u_1 y_2}{t_1}, \quad t_3 \mapsto \frac{u_2 y_3}{t_4}, \quad t_4 \mapsto \frac{y_1 u_2}{t_4}, \\ \rho_c : \gamma &\mapsto -\gamma, \quad y_2 \mapsto y_2, \quad y_3 \mapsto \frac{y_1}{y_3}, \quad t_1 \mapsto t_1, \quad t_2 \mapsto t_2, \quad t_3 \mapsto \frac{t_4}{y_3}, \quad t_4 \mapsto \frac{y_1 t_3}{y_3}\end{aligned}$$

where  $y_1, u_1, u_2$  are invariants under the action of  $G$ . The actions of  $\rho_a$ ,  $\rho_b$  and  $\rho_c$  on  $M$  are represented as matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Algorithm F2, we obtain that  $[M]^{fl}$  is not invertible (see Example 11.4 below). Hence  $L(M)^G$  is not retract  $k$ -rational. By [Yam12, Theorem 2.10],  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract  $k$ -rational.

The case (C22):  $m = 16$ . By the result of (C21),  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is not retract rational over a quadratic extension of  $k$ , hence not retract  $k$ -rational.  $\square$

**Example 11.4.** The following GAP computation confirms that  $[M]^{fl}$  is not invertible hence  $L(M)^G$  is not retract  $k$ -rational as in the cases (C16) and (C21) of the proof of Theorem 11.3 above.

```
gap> Read("FlabbyResolution.gap");

gap> v1:=[[1,0,0,0,0,0,0,0,0],[0,0,0,1,-1,0,0,0,0],[1,0,-1,0,0,0,0,0,0],
> [0,1,0,0,-1,0,0,1,0],[0,0,0,0,-1,0,0,1,0],[1,0,-1,0,0,0,1,0,0],
> [0,0,-1,0,0,1,0,0,0],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]];;
gap> v2:=[[1,0,0,0,0,0,0,0,0],[1,0,0,-1,1,0,0,0,0],[0,0,0,0,0,1,-1,0,0],
> [1,0,0,-1,0,0,0,1,0],[0,1,0,-1,0,0,0,1,0],[0,0,1,0,0,0,-1,0,1],
> [0,0,0,0,0,0,-1,0,1],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]];;
gap> IsInvertibleF(Group(v1,v2));
false

gap> v1:=[[1,0,0,0,0,0,0,0,0],[0,0,0,1,-1,0,0,0,0],[1,0,0,0,0,1,-1,0,0],
> [0,1,0,0,-1,0,0,1,0],[0,0,0,0,-1,0,0,1,0],[0,0,1,0,0,0,-1,0,1],
> [1,0,0,0,0,0,-1,0,1],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]];;
gap> v2:=[[1,0,0,0,0,0,0,0,0],[1,0,0,-1,1,0,0,0,0],[1,0,0,0,0,1,-1,0,0],
> [1,0,0,-1,0,0,0,1,0],[0,1,0,-1,0,0,0,1,0],[0,0,1,0,0,0,-1,0,1],
> [1,0,0,0,0,0,-1,0,1],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]];;
gap> v3:=[[1,0,0,0,0,0,0,0,0],[0,1,0,0,0,0,0,0,0],[1,0,-1,0,0,0,0,0,0],
> [0,0,0,1,0,0,0,0,0],[0,0,0,0,1,0,0,0,0],[0,0,-1,0,0,0,1,0,0],
> [1,0,-1,0,0,1,0,0,0],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]];;
gap> IsInvertibleF(Group(v1,v2,v3));
false
```

**Lemma 11.5.** Let  $k$  be a field of char  $k \neq 2$  and  $k(x, y, z)$  be the rational function field over  $k$  with variables  $x, y, z$ . Let  $\sigma_{a,b,c,d}$  be a  $k$ -involution on  $k(x, y, z)$  defined by

$$\sigma_{a,b,c,d} : x \mapsto -x, \quad y \mapsto \frac{-ax^2 + b}{y}, \quad z \mapsto \frac{-cx^2 + d}{z} \quad (a, b, c, d \in k^\times)$$

and  $m = [k(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) : k]$ . If one of the following conditions holds, then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational:

- (C1)  $m = 1$ ;
- (C2)  $m = 2$ , (1)  $a, b, c \in k^{\times 2}$ ; (2)  $a, b, d \in k^{\times 2}$ ; (3)  $a, c, d \in k^{\times 2}$ ; (4)  $b, c, d \in k^{\times 2}$ ;
- (C3)  $m = 2$ , (1)  $a, b, cd \in k^{\times 2}$ ; (2)  $b, d, ac \in k^{\times 2}$ ; (3)  $d, c, ab \in k^{\times 2}$ ; (4)  $c, a, bd \in k^{\times 2}$ ;
- (C5)  $m = 2$ , (1)  $a, bd, cd \in k^{\times 2}$ ; (2)  $b, cd, ac \in k^{\times 2}$ ; (3)  $d, ac, ab \in k^{\times 2}$ ; (4)  $c, ab, bd \in k^{\times 2}$ ;
- (C6)  $m = 2$ ,  $ab, ac, ad \in k^{\times 2}$ ;
- (C7)  $m = 4$ , (1)  $a, b \in k^{\times 2}$ ; (2)  $b, d \in k^{\times 2}$ ; (3)  $d, c \in k^{\times 2}$ ; (4)  $c, a \in k^{\times 2}$ ;
- (C10)  $m = 4$ , (1)  $a, bd \in k^{\times 2}$ ; (2)  $b, dc \in k^{\times 2}$ ; (3)  $d, ac \in k^{\times 2}$ ; (4)  $c, ab \in k^{\times 2}$ ;
- (5)  $a, cd \in k^{\times 2}$ ; (6)  $b, ac \in k^{\times 2}$ ; (7)  $d, ab \in k^{\times 2}$ ; (8)  $c, bd \in k^{\times 2}$ ;
- (C12)  $m = 4$ , (1)  $ab, cd \in k^{\times 2}$ ; (2)  $bd, ac \in k^{\times 2}$ ;
- (C13)  $m = 4$ , (1)  $ab, ac \in k^{\times 2}$ ; (2)  $bd, ab \in k^{\times 2}$ ; (3)  $cd, bd \in k^{\times 2}$ ; (4)  $ac, cd \in k^{\times 2}$ .

*Proof.* The case (C7):  $m = 4$ . By Theorem 11.3 (ii), we should show only the case (1)  $a, b \in k^{\times 2}$ . Take  $\alpha, \beta \in k$  with  $\alpha^2 = a$  and  $\beta^2 = b$ . The equation (19) becomes

$$\frac{(t_1 + \alpha)(t_1 - \alpha)}{(t_2 + \beta)(t_2 - \beta)} = \frac{t_3^2 - c}{t_4^2 - d}.$$

Define  $T_1 := (t_1 + \alpha)/(t_2 + \beta)$ ,  $T_2 := (t_1 - \alpha)/(t_2 - \beta)$ . Then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = k(t_1, t_2, t_3, t_4) = k(T_1, T_2, t_3, t_4) = k(T_1, t_3, t_4)$  is  $k$ -rational.

The cases (C10), (C12) and (C13):  $m = 4$ . By Theorem 11.3 (ii), we may assume that  $ab \in k^{\times 2}$ . Define  $Y := \frac{ay}{ax + \sqrt{ab}}$ . Then  $k(x, y, z) = k(x, Y, z)$  and  $\sigma_{a,b,c,d}$  acts on  $k(x, Y, z)$  by

$$\sigma_{a,b,c,d} : x \mapsto -x, \quad Y \mapsto \frac{a}{Y}, \quad z \mapsto \frac{-cx^2 + d}{z}.$$

By Theorem 11.1 (ii),  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational.

The cases (C1), (C2), (C3), (C5) and (C6). By the results of (C10), (C12) and (C13),  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational.  $\square$

**Remark 11.6.** We do not know whether  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational (resp. stably  $k$ -rational, retract  $k$ -rational) for the following cases:

- (C4)  $m = 2$ , (1)  $a, d, bc \in k^{\times 2}$ ; (2)  $b, c, ad \in k^{\times 2}$ ;
- (C8)  $m = 4$ , (1)  $a, d \in k^{\times 2}$ ; (2)  $b, c \in k^{\times 2}$ ;
- (C9)  $m = 4$ , (1)  $a, bc \in k^{\times 2}$ ; (2)  $b, ad \in k^{\times 2}$ ; (3)  $d, bc \in k^{\times 2}$ ; (4)  $c, ad \in k^{\times 2}$ ;
- (C11)  $m = 4$ , (1)  $a, bcd \in k^{\times 2}$ ; (2)  $b, acd \in k^{\times 2}$ ; (3)  $d, abc \in k^{\times 2}$ ; (4)  $c, abd \in k^{\times 2}$ ;
- (C14)  $m = 4$ ,  $ad, bc \in k^{\times 2}$ ;
- (C17)  $m = 8$ , (1)  $a \in k^{\times 2}$ ; (2)  $b \in k^{\times 2}$ ; (3)  $d \in k^{\times 2}$ ; (4)  $c \in k^{\times 2}$ .

For example, for the case of (C4)  $m = 2$ , (1)  $a, d, bc \in k^{\times 2}$ ,  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  may be obtained as follows. Let  $L = k(\alpha, \beta, \gamma, \delta)$  where  $\alpha^2 = a$ ,  $\beta^2 = b$ ,  $\gamma^2 = c$ ,  $\delta^2 = d$ . Then  $L = k(\beta)$  and  $L(x, y, z) = L(x, y', z')$  where  $y' = (\alpha x + \beta)/y$  and  $z' = (\gamma x + \delta)/z$ . We see that  $\sigma_{a,b,c,d}$  acts on  $L(x, y', z')$  by

$$\sigma_{a,b,c,d} : x \mapsto x, \quad y' \mapsto \frac{1}{y'}, \quad z' \mapsto \frac{1}{z'}.$$

Hence  $L(x, y', z')^{\langle \sigma_{a,b,c,d} \rangle} = L(y_1, y_2, y_3)$  where

$$y_1 = x^2, \quad y_2 = x \frac{1 - y'}{1 + y'} = x \frac{y - \alpha x - \beta}{y + \alpha x + \beta}, \quad y_3 = x \frac{1 - z'}{1 + z'} = x \frac{z - \gamma x - \delta}{y + \gamma x + \delta}.$$

By the assumption  $bc \in k^{\times 2}$ , there exists  $e \in k^\times$  such that  $\gamma = \beta e$ . Then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = (L(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle})^{\langle \rho_b \rangle} = L(y_1, y_2, y_3)^{\langle \rho_b \rangle}$  and

$$\rho_b : \beta \mapsto -\beta, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{y_1(\alpha y_2 + \beta)}{\alpha y_1 + \beta y_2}, \quad y_3 \mapsto \frac{\beta e y_1 + \delta y_3}{\beta e y_3 + \delta}.$$

Define

$$z_1 := \beta e \left( \frac{\beta e y_1 + \delta y_3}{\beta e y_3 + \delta} - y_3 \right) - 2\delta, \quad z_2 := e \left( \frac{\beta e y_1 + \delta y_3}{\beta e y_3 + \delta} + y_3 \right), \quad z_3 := 4be^2(\alpha y_1 + \beta y_2).$$

Then  $L(y_1, y_2, y_3) = L(z_1, z_2, z_3)$  and

$$\rho_b : \beta \mapsto -\beta, \quad z_1 \mapsto z_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto \frac{f(z_1, z_2)}{z_3}$$

where  $f(z_1, z_2) = (z_1^2 - bz_2^2 - 4d)(a(z_1^2 - bz_2^2 - 4d) + 4b^2e^2)$ . Define

$$t_1 := \frac{1}{2} \left( z_3 + \frac{f(z_1, z_2)}{z_3} \right), \quad t_2 := \frac{1}{2\beta} \left( z_3 - \frac{f(z_1, z_2)}{z_3} \right).$$

Then  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle} = k(t_1, t_2, z_1, z_2)$  where

$$t_1^2 - bt_2^2 = f(z_1, z_2) = (z_1^2 - bz_2^2 - 4d)(a(z_1^2 - bz_2^2 - 4d) + 4b^2e^2).$$

However, we do not know whether  $k(x, y, z)^{\langle \sigma_{a,b,c,d} \rangle}$  is  $k$ -rational (resp. stably  $k$ -rational, retract  $k$ -rational).

## 12. APPLICATION OF THEOREM 11.3

Let  $G$  be a finite group acting on the rational function field  $k(x_g | g \in G)$  by  $k$ -automorphisms  $h(x_g) = x_{hg}$  for any  $g, h \in G$ . We denote the fixed field  $k(x_g | g \in G)^G$  by  $k(G)$ . Noether's problem asks whether  $k(G)$  is  $k$ -rational (see a survey paper of Swan [Swa83] for Noether's problem for abelian groups). The unramified Brauer group  $Br_{v,k}(L)$  of  $L$  over  $k$  is defined to be  $Br_{v,k}(L) = \cap_R Br(R)$  where  $Br(R)$  is the image of the injective map  $Br(R) \rightarrow Br(L)$  of Brauer groups and  $R$  runs over all discrete valuation domain with  $k \subset R$  and the quotient field of  $R$  is  $L$  (cf. [Sal84b]).

Let  $F$  be an algebraically closed field with  $\gcd\{\text{char } k, |G|\} = 1$ . By [Sal90, Theorem 12], [Bog90, Theorem 1.3'],  $Br_{v,F}(F(G)) = \cap_A \text{Ker}(\text{res} : H^2(G, \mu) \rightarrow H^2(A, \mu))$  where  $\mu \subset F^\times$  denotes the subgroup of roots of unity,  $\text{res}$  is the restriction map of cohomology groups and  $A$  runs over abelian subgroups of  $G$  of rank 1 or 2. If  $G$  is a 2-group, this is valid not only over an algebraically closed field but also over a quadratically closed field  $F$  (i.e. a field of  $\text{char } F \neq 2$  satisfying  $\sqrt{a} \in F$  for any  $a \in F$ ). We denote  $Br_{v,F}(F(G))$  by  $B_0(G)$  (cf. [BMP04], [Kun10], [CHKK10]). By [Sal84b, Proposition 3.2] and [Sal87, Proposition 2.2], if  $L_1/L_2 (\supset k)$  is retract rational, then  $Br_{v,k}(L_1) \simeq Br_{v,k}(L_2)$  and hence if  $F(G)$  is retract  $F$ -rational, then  $B_0(G) = 0$ .

Saltman [Sal84b] showed that for any prime  $p$  there exists a meta-abelian  $p$ -group  $G$  of order  $p^9$  such that  $B_0(G) \neq 0$ . In particular,  $F(G)$  is not retract  $F$ -rational. Moreover, Bogomolov [Bog88] obtained that when  $\text{char } F = 0$  there exists a  $p$ -group  $G$  of order  $p^6$  such that  $B_0(G) \neq 0$  for any prime  $p$ . Recently, Moravec [Mo] proved that there exist exactly 3 groups of order  $3^5$  such that  $B_0(G) \neq 0$  by GAP computations. Hoshi, Kang and Kunyavskii [HKKu] showed that there exist exactly  $1 + \gcd\{4, p-1\} + \gcd\{3, p-1\}$  groups of order  $p^5$  such that  $B_0(G) \neq 0$  for any prime  $p \geq 5$ .

In the case where  $p = 2$ , by Chu and Kang [CK01] and Chu, Hu, Kang and Prokhorov [CHKP08], if  $G$  is a group of order  $\leq 32$ , then  $F(G)$  is  $F$ -rational. Moreover Chu, Hu, Kang and Kunyavskii [CHKK10] investigated the case where  $G$  is a group of order 64 as follows. There exist exactly 267 non-isomorphic groups of order 64.

Let  $C_n$  be the cyclic group of order  $n$ ,  $Z(G)$  be the center of the group  $G$  and  $[G, G]$  be the commutator subgroup of the group  $G$ . Let  $G(n, i)$  be the  $i$ -th group of order  $n$  in GAP [GAP].

**Theorem 12.1** ([CHKK10, Theorem 1.8]). *Let  $G$  be a group of order 64 and  $F$  be a quadratically closed field. Then the following conditions are equivalent:*

- (i)  $B_0(G) \neq 0$ ;
- (ii)  $Z(G) \simeq C_2^2$ ,  $[G, G] \simeq C_4 \times C_2$ ,  $G/[G, G] \simeq C_2^3$ ,  $G$  has no abelian subgroup of index 2, and  $G$  has no faithful 4-dimensional representation over  $\mathbb{C}$ ;
- (iii)  $G$  is isomorphic to one of the nine groups  $G(64, i)$  where  $i = 149, 150, 151, 170, 171, 172, 177, 178, 182$ .

**Theorem 12.2** ([CHKK10, Theorem 1.9]). *Let  $G$  be a group of order 64 and  $F$  be a quadratically closed field. If  $B_0(G) = 0$ , then  $F(G)$  is  $F$ -rational except possibly for groups  $G$  which is isomorphic to one of the five groups  $G(64, i)$  where  $241 \leq i \leq 245$ .*

**Theorem 12.3** ([CHKK10, Proposition 6.3]). *Let  $G = G(64, i)$  where  $241 \leq i \leq 245$ ,  $F$  be a quadratically closed field and  $H = \langle f_1, f_2 \rangle \simeq C_2 \times C_2$  act on the rational function field  $F(x_1, x_2, x_3, y_1, y_2, y_3)$  by  $F$ -automorphisms defined by*

$$\begin{aligned} f_1 : x_1 &\mapsto \frac{1}{x_1}, \quad x_2 \mapsto \frac{1}{x_1 x_3}, \quad x_3 \mapsto \frac{x_1}{x_2}, \quad y_1 \mapsto y_1, \quad y_2 \mapsto \frac{y_1}{y_2}, \quad y_3 \mapsto \frac{1}{y_1 y_3}, \\ f_2 : x_1 &\mapsto \frac{1}{x_1}, \quad x_2 \mapsto x_3, \quad x_3 \mapsto x_2, \quad y_1 \mapsto \frac{1}{y_1}, \quad y_2 \mapsto y_3, \quad y_3 \mapsto y_2. \end{aligned}$$

*Let  $L = F(x_1, x_2, x_3, y_1, y_2, y_3)^H$ . Then there is a  $F$ -injective homomorphism  $\varphi : L \rightarrow F(G)$  so that  $F(G)$  is a rational extension over  $\rho(L)$ .*

Let  $k(x_1, \dots, x_n)$  be the rational function field over  $k$  with  $n$  variables  $x_1, \dots, x_n$ . Let  $H$  be a finite subgroup of  $\text{GL}(n, \mathbb{Z})$  acting on  $k(x_1, \dots, x_n)$  by  $k$ -automorphisms

$$(20) \quad \sigma(x_j) = c_j(\sigma) \prod_{i=1}^n x_i^{a_{i,j}}, \quad \sigma = [a_{i,j}]_{1 \leq i,j \leq n} \in H, \quad c_j(\sigma) \in k^\times, \quad \text{for } 1 \leq j \leq n,$$

where  $k^\times = k \setminus \{0\}$ . This action of  $H$  on  $k(x_1, \dots, x_n)$  is said to be monomial. If  $c_j(\sigma) = 1$  for any  $\sigma \in G$  and any  $1 \leq j \leq n$ , the action said to be purely monomial. The rationality problem with respect to monomial action, i.e. whether  $k(x_1, \dots, x_n)^G$  is  $k$ -rational, is determined up to conjugacy in  $\text{GL}(n, \mathbb{Z})$  and solved affirmatively by Hajja [Haj83], [Haj87] when  $n = 2$ . For  $n = 3$ , the problem under purely monomial actions was solved affirmatively by

Hajja and Kang [HK92], [HK94] and Hoshi and Rikuna [HR08] (see also [KP10], [HK10], [Yam12] and [HKY11] for non-purely monomial case).

It is remarked in [CHKK10, page 2537] that although  $L_0 := F(x_1, x_2, x_3)^H = F(t_1, t_2, t_3)$  is  $F$ -rational, the field  $L$  as in Theorem 12.3 may be regarded as  $L = L_0(\alpha, \beta)(y_1, y_2, y_3)^H$  the function field of a 3-dimensional algebraic torus defined over  $L_0$  and split over biquadratic Galois extension  $F(x_1, x_2, x_3) = L_0(\alpha, \beta)$  of  $L_0$  for some  $\alpha, \beta$  with  $\alpha^2, \beta^2 \in L_0$ . By Theorem 1.2, the field  $L$  is not stably rational over  $L_0 = F(t_1, t_2, t_3)$ . We restate Theorem 1.2 of Kunyavskii after adopting the definition of the action of  $G$  via (20) (we should take  $G$  instead of  ${}^tG$  and hence the corresponding GAP codes may change, cf. Theorem 1.2).

**Theorem 12.4** (Kunyavskii [Kun90]). *Let  $L/k$  be a Galois extension and  $G \simeq \text{Gal}(L/k)$  be a finite subgroup of  $\text{GL}(3, \mathbb{Z})$  which acts on  $L(x_1, x_2, x_3)$  via (20). Then  $L(x_1, x_2, x_3)^G$  is not  $k$ -rational if and only if  $G$  is conjugate to one of the 15 groups which are given as in Table 10. Moreover, if  $L(x_1, x_2, x_3)^G$  is not  $k$ -rational, then it is not retract  $k$ -rational.*

Table 10:  $L(M)^G$  not retract  $k$ -rational, rank  $M = 3$ ,  $M$ : indecomposable (15 cases)

$G$ in [Kun90]	GAP code	$G$	$G$ in [Kun90]	GAP code	$G$	$G$ in [Kun90]	GAP code	$G$
$U_1$	(3, 3, 1, 4)	$C_2^2$	$U_6$	(3, 4, 7, 2)	$D_4 \times C_2$	$U_{11}$	(3, 7, 5, 2)	$S_4 \times C_2$
$U_2$	(3, 3, 3, 4)	$C_2^3$	$U_7$	(3, 7, 2, 3)	$A_4 \times C_2$	$U_{12}$	(3, 7, 5, 3)	$S_4 \times C_2$
$U_3$	(3, 4, 4, 2)	$D_4$	$U_8$	(3, 7, 3, 2)	$S_4$	$W_1$	(3, 4, 3, 2)	$C_4 \times C_2$
$U_4$	(3, 4, 6, 4)	$D_4$	$U_9$	(3, 7, 3, 3)	$S_4$	$W_2$	(3, 3, 3, 3)	$C_2^3$
$U_5$	(3, 7, 1, 3)	$A_4$	$U_{10}$	(3, 7, 4, 3)	$S_4$	$W_3$	(3, 7, 2, 2)	$S_4 \times C_2$

Let  $H = \langle f_1, f_2 \rangle \simeq C_2 \times C_2$  act on  $k(x_1, x_2, x_3, y_1, y_2, y_3)$  by  $k$ -automorphisms as the same in Theorem 12.3. The purely monomial actions of  $H$  on  $k(x_1, x_2, x_3)$  and on  $k(y_1, y_2, y_3)$  correspond to the same conjugacy class of the GAP code (3, 3, 1, 4). Indeed, if we define  $(X_1, X_2, X_3, Y_1, Y_2, Y_3) := (x_1/x_2, 1/(x_1x_3), x_3, 1/(y_1y_3), y_1/y_2, y_3)$ , then  $k(x_1, x_2, x_3, y_1, y_2, y_3) = k(X_1, X_2, X_3, Y_1, Y_2, Y_3)$  and the action of  $H$  on  $k(X, 1, X_2, X_3, Y_1, Y_2, Y_3)$  is given by

$$\begin{aligned} \sigma_1 : X_1 &\mapsto X_3, X_2 \mapsto \frac{1}{X_1X_2X_3}, X_3 \mapsto X_1, Y_1 \mapsto Y_3, Y_2 \mapsto \frac{1}{Y_1Y_2Y_3}, Y_3 \mapsto Y_1, \\ \sigma_2 : X_1 &\mapsto X_2, X_2 \mapsto X_1, X_3 \mapsto \frac{1}{X_1X_2X_3}, Y_1 \mapsto Y_2, Y_2 \mapsto Y_1, Y_3 \mapsto \frac{1}{Y_1Y_2Y_3}. \end{aligned}$$

We define

$$s_1 := \frac{1 + X_1X_3}{1 - X_1X_3}, s_2 := \frac{1 + X_2X_3}{1 - X_2X_3}, s_3 := X_3, s_4 := \frac{1 + Y_1Y_3}{1 - Y_1Y_3}, s_5 := \frac{1 + Y_2Y_3}{1 - Y_2Y_3}, s_6 := Y_3.$$

Then  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3) = k(s_1, \dots, s_6)$  and the actions of  $\sigma_1$  and  $\sigma_2$  on  $k(s_1, \dots, s_6)$  are given by

$$\begin{aligned} \sigma_1 : s_1 &\mapsto s_1, s_2 \mapsto -s_2, s_3 \mapsto \frac{s_1 - 1}{s_3(s_1 + 1)}, s_4 \mapsto s_4, s_5 \mapsto -s_5, s_6 \mapsto \frac{s_4 - 1}{s_6(s_4 + 1)}, \\ \sigma_2 : s_1 &\mapsto -s_1, s_2 \mapsto -s_2, s_3 \mapsto \frac{s_3(s_1 + 1)(s_2 + 1)}{(s_1 - 1)(s_2 - 1)}, s_4 \mapsto -s_4, s_5 \mapsto -s_5, s_6 \mapsto \frac{s_6(s_4 + 1)(s_5 + 1)}{(s_4 - 1)(s_5 - 1)}. \end{aligned}$$

We also define

$$t_1 := s_1s_2, t_2 := \frac{(s_1s_2 + 1)s_3}{(s_1 - 1)(s_2 - 1)}, t_3 := \frac{(s_4s_5 + 1)s_6}{(s_4 - 1)(s_5 - 1)}, t_4 := s_2^2, t_5 := \frac{s_1}{s_4}, t_6 := \frac{s_2}{s_5}.$$

Then we have  $k(s_1, \dots, s_6) = k(t_1, t_2, t_3, s_2, t_5, t_6)$  and  $\sigma_2 : s_2 \mapsto -s_2, t_i \mapsto t_i, 1 \leq i \leq 6$ . It follows that  $k(x_1, x_2, x_3, y_1, y_2, y_3)^{\langle \sigma_2 \rangle} = k(t_1, \dots, t_6)$  and the action of  $\sigma_1$  on  $k(t_1, \dots, t_6)$  is given by

$$\sigma_1 : t_1 \mapsto -t_1, t_2 \mapsto \frac{(t_1^2 - 1)t_4}{t_2(t_1^2 - t_4)(t_4 - 1)}, t_3 \mapsto \frac{t_4(t_5^2t_6^2 - t_1^2)}{t_3(t_6^2 - t_4)(t_1^2 - t_4t_5^2)}, t_4 \mapsto t_4, t_5 \mapsto t_5, t_6 \mapsto t_6.$$

Define

$$u_1 := \frac{t_1}{t_4}, u_2 := \frac{t_1 + 1}{t_2}, u_3 := \frac{t_5t_6 + t_1}{t_3}, u_4 := t_4, u_5 := t_5, u_6 := t_6.$$

Then we get  $k(t_1, \dots, t_6) = k(u_1, \dots, u_6)$  and the action of  $\sigma_1$  on  $k(u_1, \dots, u_6)$  is given by

$$\begin{aligned} (21) \quad \sigma_1 : u_1 &\mapsto -u_1, u_2 \mapsto -\frac{(u_4 - 1)(u_1^2u_4 - 1)}{u_2}, u_3 \mapsto -\frac{(u_4 - u_6^2)(u_1^2u_4 - u_5^2)}{u_3}, \\ u_4 &\mapsto u_4, u_5 \mapsto u_5, u_6 \mapsto u_6. \end{aligned}$$

As a consequence of Theorem 11.3, we have:

**Proposition 12.5.** *Let  $k$  be a field of char  $k \neq 2$  and  $\langle \sigma_1, \sigma_2 \rangle \simeq C_2 \times C_2$  act on  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)$  by  $k$ -automorphisms*

$$\begin{aligned}\sigma_1 : X_1 &\mapsto X_3, X_2 \mapsto \frac{1}{X_1 X_2 X_3}, X_3 \mapsto X_1, Y_1 \mapsto Y_3, Y_2 \mapsto \frac{1}{Y_1 Y_2 Y_3}, Y_3 \mapsto Y_1, \\ \sigma_2 : X_1 &\mapsto X_2, X_2 \mapsto X_1, X_3 \mapsto \frac{1}{X_1 X_2 X_3}, Y_1 \mapsto Y_2, Y_2 \mapsto Y_1, Y_3 \mapsto \frac{1}{Y_1 Y_2 Y_3}.\end{aligned}$$

*There exist algebraically independent elements  $u_4, u_5, u_6 \in k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle}$  over  $k$  and algebraically dependent elements  $z_1, z_2, z_3, z_4 \in k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle}$  over  $k(u_4, u_5, u_6)$  which satisfy the following conditions:*

(i)  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle} = k(z_1, z_2, z_3, z_4, u_4, u_5, u_6)$  and

$$(z_1^2 - a)(z_4^2 - d) = (z_2^2 - b)(z_3^2 - c)$$

where  $a = u_4(u_4 - 1), b = u_4 - 1, c = u_4(u_4 - u_6^2), d = u_5^2(u_4 - u_6^2)$ ;

(ii)  $k(z_1, z_2, z_3, z_4, u_4, u_5, u_6)$  is not retract rational over  $k(u_4, u_5, u_6)$ .

*Proof.* If we take a field  $k(u_4, u_5, u_6)$  as the base field, the assertion (i) follows from (21) and Theorem 11.3 (i). For (ii), we see that  $abcd \in k(u_4, u_5, u_6)^{\times 2}$  and  $m = [k(u_4, u_5, u_6)(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) : k(u_4, u_5, u_6)] = 8$ . Thus (ii) follows from Theorem 11.3 (iii).  $\square$

Furthermore, we define

$$v_1 := u_1, v_2 := \frac{u_2}{1 - u_1^2 u_4}, v_3 := \frac{(u_1^2 - 1)u_3}{1 - u_1^2 u_4}, v_4 := \frac{u_4 - 1}{1 - u_1^2 u_4}, v_5 := u_5, v_6 := u_6.$$

Then  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_2 \rangle} = k(u_1, \dots, u_6) = k(v_1, \dots, v_6)$  and

$$\sigma_1 : v_1 \mapsto -v_1, v_2 \mapsto \frac{v_4}{v_2}, v_3 \mapsto -\frac{(v_1^2 v_4 v_5^2 - v_1^2 v_4 - v_1^2 + v_5^2)(v_1^2 v_4 v_6^2 + v_6^2 - v_4 - 1)}{v_3}, v_4 \mapsto v_4, v_5 \mapsto v_5, v_6 \mapsto v_6.$$

We also put

$$w_1 := v_1, w_2 := v_3, w_3 := \frac{1}{2} \left( v_2 + \frac{v_4}{v_2} \right), w_4 := \frac{1}{2v_1} \left( v_2 - \frac{v_4}{v_2} \right), w_5 := v_5, w_6 := v_6.$$

Then  $k(v_1, \dots, v_6) = k(w_1, \dots, w_6)$  and

$$\sigma_1 : w_1 \mapsto -w_1, w_2 \mapsto -\frac{(w_4^2(w_5^2 - 1)w_1^4 + (w_3^2 - w_3^2 w_5^2 + 1)w_1^2 - w_5^2)(w_4^2 w_6^2 w_1^4 - (w_4^2 + w_3^2 w_6^2)w_1^2 + w_3^2 - w_6^2 + 1)}{w_2},$$

$$w_3 \mapsto w_3, w_4 \mapsto w_4, w_5 \mapsto w_5, w_6 \mapsto w_6.$$

Define  $t = w_1^2$ . Then  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle} = k(u, v, t, w_3, w_4, w_5, w_6)$  with the relation

$$u^2 - tv^2 = -(w_4^2(w_5^2 - 1)t^2 + (w_3^2 - w_3^2 w_5^2 + 1)t - w_5^2)(w_4^2 w_6^2 t^2 - (w_4^2 + w_3^2 w_6^2)t + w_3^2 - w_6^2 + 1)$$

(cf. [HKKi, Section 6]).

We do not know whether  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle}$  is  $k$ -rational (resp. stably  $k$ -rational, retract  $k$ -rational). We may also obtain the following description of  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle}$  although the proof is omitted.

**Theorem 12.6.** *The field  $k(X_1, X_2, X_3, Y_1, Y_2, Y_3)^{\langle \sigma_1, \sigma_2 \rangle} = k(m_0, \dots, m_6)$  where*

$$m_0^2 = (4m_3 + m_3 m_4^2 + m_4^2)(m_3 - m_5^2 + 1)(m_1^2 m_3 + m_6^2 - 1)(4m_3 + m_1^2 m_2^2 m_3 + m_2^2 m_6^2).$$



13. TABLES FOR A BIRATIONAL CLASSIFICATION OF THE ALGEBRAIC  $k$ -TORI OF DIMENSION 5

Table 11:  $L(M)^G$  not stably but retract  $k$ -rational,  $M = M_1 \oplus M_2$ ,  
rank  $M_1 = 4$ , rank  $M_2 = 1$ ,  $M_i$  : indecomposable (25 cases)

CARAT	$G$	CARAT	$G$	CARAT	$G$	CARAT	$G$	CARAT	$G$
(5,692,1)	$C_3 \rtimes C_8$	(5,914,1)	$S_5$	(5,918,5)	$F_{20}$	(5,924,1)	$C_2 \times S_5$	(5,928,1)	$C_2 \times F_{20}$
(5,693,1)	$C_3 \rtimes C_8$	(5,916,1)	$S_5$	(5,919,1)	$C_2 \times S_5$	(5,925,1)	$C_2 \times S_5$	(5,928,3)	$C_2 \times F_{20}$
(5,736,1)	$C_2 \times (C_3 \rtimes C_8)$	(5,917,1)	$F_{20}$	(5,921,1)	$C_2 \times S_5$	(5,926,1)	$C_2 \times F_{20}$	(5,929,1)	$C_2 \times F_{20}$
(5,911,1)	$S_5$	(5,917,5)	$F_{20}$	(5,922,1)	$C_2 \times S_5$	(5,926,3)	$C_2 \times F_{20}$	(5,930,1)	$C_2^2 \times S_5$
(5,912,1)	$S_5$	(5,918,1)	$F_{20}$	(5,923,1)	$C_2 \times S_5$	(5,927,1)	$C_2 \times F_{20}$	(5,932,1)	$C_2^2 \times F_{20}$

Table 12:  $L(M)^G$  not retract  $k$ -rational,  $M = M_1 \oplus M_2 \oplus M_3$ ,  
rank  $M_1 = 3$ , rank  $M_2 = \text{rank } M_3 = 1$ ,  $M_i$  : indecomposable (245 cases)

CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT
(5,11,4)	(5,24,8)	(5,32,8)	(5,55,2)	(5,72,2)	(5,81,3)	(5,94,14)	(5,510,1)	(5,524,1)	(5,538,9)
(5,12,4)	(5,24,10)	(5,32,10)	(5,55,3)	(5,72,21)	(5,81,6)	(5,95,2)	(5,510,4)	(5,524,9)	(5,539,1)
(5,12,5)	(5,24,20)	(5,32,26)	(5,56,2)	(5,72,22)	(5,81,29)	(5,96,2)	(5,511,1)	(5,525,1)	(5,539,8)
(5,14,4)	(5,25,3)	(5,40,2)	(5,56,3)	(5,73,2)	(5,82,3)	(5,97,2)	(5,511,4)	(5,526,1)	(5,540,1)
(5,16,4)	(5,25,5)	(5,44,2)	(5,59,3)	(5,73,21)	(5,82,6)	(5,98,6)	(5,512,1)	(5,526,8)	(5,540,9)
(5,16,5)	(5,25,9)	(5,45,2)	(5,60,2)	(5,73,22)	(5,83,2)	(5,99,2)	(5,512,4)	(5,527,1)	(5,541,1)
(5,17,4)	(5,25,10)	(5,46,2)	(5,60,10)	(5,74,2)	(5,83,3)	(5,99,3)	(5,513,1)	(5,527,9)	(5,541,9)
(5,17,5)	(5,26,3)	(5,47,2)	(5,61,2)	(5,75,2)	(5,84,3)	(5,99,32)	(5,514,1)	(5,528,1)	(5,542,1)
(5,17,7)	(5,26,5)	(5,47,3)	(5,62,10)	(5,75,21)	(5,85,3)	(5,100,2)	(5,514,4)	(5,528,9)	(5,542,9)
(5,17,12)	(5,26,8)	(5,48,2)	(5,63,2)	(5,76,2)	(5,86,2)	(5,100,3)	(5,515,1)	(5,529,1)	(5,543,1)
(5,18,7)	(5,26,12)	(5,48,3)	(5,63,9)	(5,76,6)	(5,87,2)	(5,101,2)	(5,515,4)	(5,529,9)	(5,544,1)
(5,19,5)	(5,26,22)	(5,48,9)	(5,63,10)	(5,76,28)	(5,88,2)	(5,102,2)	(5,516,1)	(5,530,1)	(5,544,8)
(5,20,4)	(5,27,3)	(5,48,10)	(5,64,3)	(5,76,29)	(5,88,3)	(5,103,6)	(5,516,4)	(5,530,9)	(5,545,1)
(5,20,5)	(5,27,6)	(5,49,2)	(5,65,3)	(5,76,32)	(5,88,21)	(5,502,6)	(5,517,1)	(5,531,1)	(5,545,8)
(5,20,7)	(5,28,3)	(5,49,3)	(5,65,9)	(5,77,2)	(5,88,22)	(5,503,1)	(5,517,4)	(5,532,1)	(5,546,1)
(5,20,12)	(5,28,9)	(5,49,10)	(5,65,10)	(5,77,6)	(5,89,2)	(5,503,4)	(5,518,1)	(5,533,1)	(5,546,8)
(5,21,4)	(5,29,3)	(5,50,2)	(5,66,2)	(5,78,2)	(5,89,3)	(5,504,1)	(5,518,8)	(5,533,9)	(5,547,1)
(5,21,5)	(5,29,6)	(5,50,3)	(5,66,3)	(5,78,3)	(5,90,2)	(5,504,4)	(5,519,1)	(5,534,1)	(5,547,8)
(5,21,7)	(5,30,5)	(5,51,3)	(5,67,2)	(5,78,22)	(5,90,21)	(5,505,1)	(5,519,16)	(5,534,8)	(5,548,1)
(5,22,5)	(5,30,10)	(5,52,2)	(5,68,2)	(5,79,2)	(5,90,24)	(5,506,1)	(5,520,16)	(5,535,1)	(5,548,9)
(5,22,12)	(5,31,4)	(5,52,3)	(5,69,3)	(5,79,3)	(5,91,2)	(5,507,1)	(5,521,16)	(5,536,1)	
(5,23,7)	(5,31,6)	(5,52,9)	(5,70,3)	(5,79,21)	(5,91,3)	(5,507,4)	(5,522,1)	(5,536,9)	
(5,23,15)	(5,31,7)	(5,53,2)	(5,71,2)	(5,79,22)	(5,92,2)	(5,508,1)	(5,522,16)	(5,537,1)	
(5,24,5)	(5,31,12)	(5,53,3)	(5,71,3)	(5,80,2)	(5,93,2)	(5,508,4)	(5,523,1)	(5,537,9)	
(5,24,7)	(5,31,19)	(5,54,2)	(5,71,6)	(5,80,3)	(5,94,2)	(5,509,1)	(5,523,9)	(5,538,1)	

Table 13:  $L(M)^G$  not retract  $k$ -rational,  $M = M_1 \oplus M_2$ ,  
rank  $M_1 = 3$ , rank  $M_2 = 2$ ,  $M_i$  : indecomposable (849 cases)

CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT
(5,14,8)	(5,73,35)	(5,95,10)	(5,140,16)	(5,240,6)	(5,299,4)	(5,349,2)	(5,528,12)	(5,569,8)	(5,603,3)
(5,16,8)	(5,74,12)	(5,96,5)	(5,140,20)	(5,240,14)	(5,302,2)	(5,349,4)	(5,529,2)	(5,569,9)	(5,603,4)
(5,16,9)	(5,75,12)	(5,96,8)	(5,140,22)	(5,240,15)	(5,302,10)	(5,350,2)	(5,529,12)	(5,570,5)	(5,603,6)
(5,17,11)	(5,75,31)	(5,96,10)	(5,141,4)	(5,241,4)	(5,303,2)	(5,350,4)	(5,530,2)	(5,570,6)	(5,604,1)
(5,17,16)	(5,76,11)	(5,97,5)	(5,141,10)	(5,241,5)	(5,303,6)	(5,350,6)	(5,530,12)	(5,570,8)	(5,604,5)
(5,18,19)	(5,76,15)	(5,97,8)	(5,141,14)	(5,241,9)	(5,304,2)	(5,350,8)	(5,531,2)	(5,570,9)	(5,605,1)
(5,20,11)	(5,76,20)	(5,97,10)	(5,141,16)	(5,241,11)	(5,304,10)	(5,351,2)	(5,532,2)	(5,571,5)	(5,606,1)
(5,20,16)	(5,76,21)	(5,98,20)	(5,141,20)	(5,242,4)	(5,305,2)	(5,351,4)	(5,533,2)	(5,571,8)	(5,606,5)
(5,21,11)	(5,76,37)	(5,99,15)	(5,141,22)	(5,242,14)	(5,305,6)	(5,352,2)	(5,533,12)	(5,572,3)	(5,607,1)
(5,21,16)	(5,76,41)	(5,99,18)	(5,142,2)	(5,244,4)	(5,305,10)	(5,353,2)	(5,534,2)	(5,573,3)	(5,607,5)
(5,23,14)	(5,76,44)	(5,99,47)	(5,142,4)	(5,244,6)	(5,305,12)	(5,353,4)	(5,534,10)	(5,573,4)	(5,608,1)
(5,23,26)	(5,76,46)	(5,100,11)	(5,142,11)	(5,244,14)	(5,306,4)	(5,354,2)	(5,535,2)	(5,574,5)	(5,608,5)
(5,24,16)	(5,76,47)	(5,100,15)	(5,142,13)	(5,244,15)	(5,306,10)	(5,355,2)	(5,536,2)	(5,574,6)	(5,608,7)
(5,24,19)	(5,77,11)	(5,100,18)	(5,142,17)	(5,246,5)	(5,309,2)	(5,357,4)	(5,536,12)	(5,574,8)	(5,608,10)
(5,24,25)	(5,77,15)	(5,101,12)	(5,142,19)	(5,246,7)	(5,309,4)	(5,359,2)	(5,537,2)	(5,574,9)	(5,609,1)
(5,25,15)	(5,77,20)	(5,102,8)	(5,142,23)	(5,246,17)	(5,309,6)	(5,359,6)	(5,537,12)	(5,575,3)	(5,609,3)
(5,25,18)	(5,77,21)	(5,102,12)	(5,142,25)	(5,246,18)	(5,309,8)	(5,360,2)	(5,538,2)	(5,575,4)	(5,610,1)
(5,25,24)	(5,78,8)	(5,103,20)	(5,143,2)	(5,247,3)	(5,310,2)	(5,360,4)	(5,538,12)	(5,576,3)	(5,610,3)
(5,25,26)	(5,78,11)	(5,104,2)	(5,143,4)	(5,247,7)	(5,310,6)	(5,360,6)	(5,539,2)	(5,577,3)	(5,611,1)
(5,26,17)	(5,78,14)	(5,105,2)	(5,143,8)	(5,248,3)	(5,311,2)	(5,360,8)	(5,539,10)	(5,577,4)	(5,611,3)
(5,26,20)	(5,78,27)	(5,105,4)	(5,143,10)	(5,248,4)	(5,311,4)	(5,361,2)	(5,540,2)	(5,578,3)	(5,612,1)
(5,26,25)	(5,78,33)	(5,105,8)	(5,143,14)	(5,248,7)	(5,312,2)	(5,361,4)	(5,540,12)	(5,578,4)	(5,612,3)
(5,26,35)	(5,79,8)	(5,107,4)	(5,143,16)	(5,248,8)	(5,312,6)	(5,361,6)	(5,541,2)	(5,579,3)	(5,612,4)
(5,26,37)	(5,79,11)	(5,109,2)	(5,143,20)	(5,249,4)	(5,315,4)	(5,361,8)	(5,541,12)	(5,579,4)	(5,612,6)
(5,27,9)	(5,79,14)	(5,109,4)	(5,143,22)	(5,249,5)	(5,315,8)	(5,362,2)	(5,542,2)	(5,580,1)	(5,613,1)
(5,27,13)	(5,79,16)	(5,109,8)	(5,144,4)	(5,249,9)	(5,316,2)	(5,362,4)	(5,542,12)	(5,581,1)	(5,613,3)
(5,28,18)	(5,79,27)	(5,109,10)	(5,144,8)	(5,249,10)	(5,317,2)	(5,363,2)	(5,543,2)	(5,581,3)	(5,613,4)
(5,28,24)	(5,79,30)	(5,110,4)	(5,144,10)	(5,251,4)	(5,317,4)	(5,363,4)	(5,544,2)	(5,582,1)	(5,613,6)
(5,29,9)	(5,79,33)	(5,111,2)	(5,145,2)	(5,251,5)	(5,318,2)	(5,364,2)	(5,544,10)	(5,582,3)	(5,614,1)
(5,29,13)	(5,79,35)	(5,111,8)	(5,145,4)	(5,251,9)	(5,319,2)	(5,364,4)	(5,545,2)	(5,583,1)	(5,614,3)
(5,30,18)	(5,80,8)	(5,111,10)	(5,145,8)	(5,251,10)	(5,320,2)	(5,364,6)	(5,545,10)	(5,583,3)	(5,615,1)
(5,30,26)	(5,80,11)	(5,112,2)	(5,145,10)	(5,253,4)	(5,322,2)	(5,364,8)	(5,546,2)	(5,584,1)	(5,616,1)
(5,31,17)	(5,80,14)	(5,113,2)	(5,146,4)	(5,254,5)	(5,323,2)	(5,365,2)	(5,546,10)	(5,584,3)	(5,616,3)
(5,31,23)	(5,81,18)	(5,115,2)	(5,146,8)	(5,254,6)	(5,323,4)	(5,365,6)	(5,547,2)	(5,585,1)	(5,617,1)
(5,31,30)	(5,81,20)	(5,116,2)	(5,146,10)	(5,256,3)	(5,324,2)	(5,368,2)	(5,547,10)	(5,585,3)	(5,617,3)
(5,31,38)	(5,81,44)	(5,116,4)	(5,147,2)	(5,256,4)	(5,325,2)	(5,368,4)	(5,548,2)	(5,586,1)	(5,617,4)
(5,31,42)	(5,82,18)	(5,116,11)	(5,148,2)	(5,257,4)	(5,326,2)	(5,369,2)	(5,548,12)	(5,587,1)	(5,617,6)
(5,32,17)	(5,82,20)	(5,116,17)	(5,148,7)	(5,257,5)	(5,326,4)	(5,369,4)	(5,549,3)	(5,587,3)	(5,618,1)
(5,32,24)	(5,83,8)	(5,116,19)	(5,149,2)	(5,257,9)	(5,327,2)	(5,369,6)	(5,549,4)	(5,588,1)	(5,618,3)
(5,32,50)	(5,83,12)	(5,116,23)	(5,149,10)	(5,257,11)	(5,328,2)	(5,369,8)	(5,550,5)	(5,588,3)	(5,618,4)
(5,35,2)	(5,83,14)	(5,118,12)	(5,149,12)	(5,258,3)	(5,328,4)	(5,370,2)	(5,550,6)	(5,589,1)	(5,618,6)
(5,36,2)	(5,83,16)	(5,118,16)	(5,149,16)	(5,258,4)	(5,331,2)	(5,371,2)	(5,550,8)	(5,589,3)	(5,619,1)
(5,36,5)	(5,84,14)	(5,119,2)	(5,150,2)	(5,258,7)	(5,332,2)	(5,372,2)	(5,550,9)	(5,589,4)	(5,619,3)
(5,38,5)	(5,85,8)	(5,119,7)	(5,150,4)	(5,258,8)	(5,333,2)	(5,372,4)	(5,551,5)	(5,589,6)	(5,620,1)
(5,38,8)	(5,85,14)	(5,120,2)	(5,151,2)	(5,259,3)	(5,333,4)	(5,373,2)	(5,551,6)	(5,590,1)	(5,621,1)
(5,39,2)	(5,85,16)	(5,120,10)	(5,151,7)	(5,260,3)	(5,334,2)	(5,373,4)	(5,551,8)	(5,591,1)	(5,621,5)
(5,39,4)	(5,86,5)	(5,122,2)	(5,151,9)	(5,260,4)	(5,334,4)	(5,373,6)	(5,551,9)	(5,591,3)	(5,622,1)
(5,47,7)	(5,86,8)	(5,122,13)	(5,151,13)	(5,261,3)	(5,335,2)	(5,373,8)	(5,552,5)	(5,591,4)	(5,622,5)
(5,48,7)	(5,87,5)	(5,123,2)	(5,152,2)	(5,261,4)	(5,335,4)	(5,374,2)	(5,552,6)	(5,591,6)	(5,623,1)
(5,48,14)	(5,87,8)	(5,123,4)	(5,153,2)	(5,261,7)	(5,336,2)	(5,374,4)	(5,552,8)	(5,592,1)	(5,623,5)
(5,49,7)	(5,87,10)	(5,123,11)	(5,153,4)	(5,261,8)	(5,336,4)	(5,376,4)	(5,552,9)	(5,592,3)	(5,623,7)
(5,49,14)	(5,88,8)	(5,123,13)	(5,153,8)	(5,262,4)	(5,336,6)	(5,377,2)	(5,553,3)	(5,592,4)	(5,623,10)
(5,50,7)	(5,88,12)	(5,124,2)	(5,153,10)	(5,262,5)	(5,336,10)	(5,377,4)	(5,553,4)	(5,592,6)	(5,624,1)
(5,51,7)	(5,88,14)	(5,124,4)	(5,154,2)	(5,264,4)	(5,337,2)	(5,378,2)	(5,554,3)	(5,593,1)	(5,624,5)
(5,52,7)	(5,88,16)	(5,125,2)	(5,154,7)	(5,264,5)	(5,337,6)	(5,378,4)	(5,554,4)	(5,593,3)	(5,624,7)
(5,53,7)	(5,88,27)	(5,125,4)	(5,154,9)	(5,265,4)	(5,338,2)	(5,379,2)	(5,555,3)	(5,593,4)	(5,624,10)
(5,55,7)	(5,88,31)	(5,127,2)	(5,154,13)	(5,265,5)	(5,338,4)	(5,380,2)	(5,556,3)	(5,593,6)	(5,625,1)
(5,56,7)	(5,88,33)	(5,127,4)	(5,155,2)	(5,265,9)	(5,339,2)	(5,380,4)	(5,556,4)	(5,594,1)	(5,626,1)
(5,59,7)	(5,88,35)	(5,127,8)	(5,157,2)	(5,265,10)	(5,339,4)	(5,380,6)	(5,557,3)	(5,594,3)	(5,626,3)
(5,60,14)	(5,89,8)	(5,127,10)	(5,157,4)	(5,266,3)	(5,340,2)	(5,380,8)	(5,557,4)	(5,594,4)	(5,627,1)
(5,63,14)	(5,89,12)	(5,128,2)	(5,157,8)	(5,266,4)	(5,340,4)	(5,381,2)	(5,558,3)	(5,594,6)	(5,627,3)
(5,64,7)	(5,89,14)	(5,128,8)	(5,158,2)	(5,273,2)	(5,340,6)	(5,381,4)	(5,558,4)	(5,595,1)	(5,628,1)
(5,65,7)	(5,89,16)	(5,128,10)	(5,158,4)	(5,274,2)	(5,340,10)	(5,382,2)	(5,559,3)	(5,595,3)	(5,628,3)
(5,65,14)	(5,90,8)	(5,129,4)	(5,158,8)	(5,275,2)	(5,342,2)	(5,383,2)	(5,559,4)	(5,595,4)	(5,629,1)
(5,66,7)	(5,90,12)	(5,130,2)	(5,158,10)	(5,276,2)	(5,342,6)	(5,384,2)	(5,560,3)	(5,595,6)	(5,630,1)
(5,69,7)	(5,90,15)	(5,130,4)	(5,159,2)	(5,278,2)	(5,343,2)	(5,384,4)	(5,561,3)	(5,596,1)	(5,630,3)
(5,70,7)	(5,90,27)	(5,131,2)	(5,159,4)	(5,279,2)	(5,343,4)	(5,385,2)	(5,561,4)	(5,596,3)	(5,631,1)
(5,71,11)	(5,90,31)	(5,131,7)	(5,160,2)	(5,280,2)	(5,343,6)	(5,518,2)	(5,562,3)	(5,597,1)	(5,631,3)
(5,71,15)	(5,90,33)	(5,132,2)	(5,224,4)	(5,280,4)	(5,343,8)	(5,518,10)	(5,562,4)	(5,597,3)	(5,632,1)
(5,71,18)	(5,90,34)	(5,132,7)	(5,227,5)	(5,281,2)	(5,344,2)	(5,519,4)	(5,563,3)	(5,597,4)	(5,633,1)
(5,71,20)	(5,91,8)	(5,132,9)	(5,228,3)	(5,282,2)	(5,344,6)	(5,519,17)	(5,564,3)	(5,597,6)	(5,634,1)
(5,71,21)	(5,91,11)	(5,132,13)	(5,232,4)	(5,285,2)	(5,345,2)	(5,520,17)	(5,564,4)	(5,598,1)	(5,634,3)
(5,72,8)	(5,91,14)	(5,134,2)	(5,232,9)	(5,288,2)	(5,345,4)	(5,521,17)	(5,565,5)	(5,598,3)	(5,635,1)
(5,72,12)	(5,91,16)	(5,134,7)	(5,233,4)	(5,289,2)	(5,346,2)	(5,522,4)	(5,565,6)	(5,599,1)	(5,635,3)
(5,72,16)	(5,92,12)	(5,135,2)	(5,233,6)	(5,290,2)	(5,346,4)	(5,522,17)	(5,565,8)	(5,599,3)	(5,635,4)
(5,72,27)	(5,93,5)	(5,136,2)	(5,234,5)	(5,290,4)	(5,346,6)	(5,523,2)	(5,565,9)	(5,599,4)	(5,635,6)
(5,72,31)	(5,93,8)	(5,136,10)	(5,234,7)	(5,291,2)	(5,346,8)	(5,523,12)	(5,566,3)	(5,599,6)	(5,636,1)
(5,72,33)	(5,94,5)	(5,136,12)	(5,235,3)	(5,291,4)	(5,347,2)	(5,524,2)	(5,567,3)	(5,600,1)	(5,636,3)
(5,72,35)	(5,94,8)	(5,136,16)	(5,235,4)	(5,294,2)	(5,347,4)	(5,524,12)	(5,567,4)	(5,600,3)	(5,636,4)
(5,73,8)	(5,94,10)	(5,139,2)	(5,236,3)	(5,294,4)	(5,347,6)	(5,525,2)	(5,568,5)	(5,600,4)	(5,636,6)
(5,73,12)	(5,94,17)	(5,139,8)	(5,236,4)	(5,295,2)	(5,347,8)	(5,526,2)	(5,568,6)	(5,600,6)	(5,637,1)
(5,73,16)	(5,94,20)	(5,139,10)	(5,237,3)	(5,295,4)	(5,348,2)	(5,526,10)	(5,568,8)	(5,601,1)	(5,637,3)
(5,73,27)	(5,94,22)	(5,140,4)	(5,239,4)	(5,296,2)	(5,348,4)	(5,527,2)	(5,568,9)	(5,602,1)	(5,638,1)
(5,73,31)	(5,95,5)	(5,140,10)	(5,239,5)	(5,296,4)	(5,348,6)	(5,527,12)	(5,569,5)	(5,602,3)	(5,638,5)
(5,73,33)	(5,95,8)	(5,140,14)	(5,240,4)	(5,299,2)	(5,348,8)	(5,528,2)	(5,569,6)	(5,603,1)	

Table 14:  $L(M)^G$  not retract  $k$ -rational,  
 $M = M_1 \oplus M_2$ , rank  $M_1 = 4$ , rank  $M_2 = 1$ ,  $M_i$  : indecomposable (768 cases)

CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT
(5,17,6)	(5,49,13)	(5,78,10)	(5,97,4)	(5,133,17)	(5,430,2)	(5,528,4)	(5,662,1)	(5,726,5)	(5,785,1)
(5,17,10)	(5,50,5)	(5,78,13)	(5,97,7)	(5,134,3)	(5,431,2)	(5,528,10)	(5,663,1)	(5,727,1)	(5,785,2)
(5,17,13)	(5,50,6)	(5,78,26)	(5,98,14)	(5,134,8)	(5,432,2)	(5,528,11)	(5,664,1)	(5,727,5)	(5,786,1)
(5,17,15)	(5,51,6)	(5,78,29)	(5,98,17)	(5,135,3)	(5,433,2)	(5,529,4)	(5,665,3)	(5,728,1)	(5,786,2)
(5,18,18)	(5,52,5)	(5,78,32)	(5,99,5)	(5,136,3)	(5,434,3)	(5,529,10)	(5,666,3)	(5,728,2)	(5,797,3)
(5,18,21)	(5,52,6)	(5,79,5)	(5,99,13)	(5,136,11)	(5,435,3)	(5,529,11)	(5,667,3)	(5,729,1)	(5,798,2)
(5,19,10)	(5,52,13)	(5,79,7)	(5,99,14)	(5,136,17)	(5,436,3)	(5,530,5)	(5,668,1)	(5,729,2)	(5,801,4)
(5,20,6)	(5,53,5)	(5,79,10)	(5,99,17)	(5,137,3)	(5,444,6)	(5,530,10)	(5,669,1)	(5,730,1)	(5,801,5)
(5,20,10)	(5,53,6)	(5,79,13)	(5,99,42)	(5,138,3)	(5,445,6)	(5,530,11)	(5,670,1)	(5,731,1)	(5,805,3)
(5,20,13)	(5,54,6)	(5,79,24)	(5,99,46)	(5,138,9)	(5,446,6)	(5,531,5)	(5,671,1)	(5,732,1)	(5,806,3)
(5,20,15)	(5,55,5)	(5,79,26)	(5,100,5)	(5,139,3)	(5,447,6)	(5,531,11)	(5,672,1)	(5,733,1)	(5,808,2)
(5,21,6)	(5,55,6)	(5,79,29)	(5,100,13)	(5,139,9)	(5,448,2)	(5,532,4)	(5,673,1)	(5,734,1)	(5,809,2)
(5,21,10)	(5,56,5)	(5,79,32)	(5,100,14)	(5,140,3)	(5,449,2)	(5,532,9)	(5,674,1)	(5,735,1)	(5,810,3)
(5,21,15)	(5,56,6)	(5,80,5)	(5,100,17)	(5,140,9)	(5,450,2)	(5,533,10)	(5,675,1)	(5,737,1)	(5,810,4)
(5,22,10)	(5,59,6)	(5,80,7)	(5,101,4)	(5,140,15)	(5,450,5)	(5,533,11)	(5,676,1)	(5,738,1)	(5,811,2)
(5,22,15)	(5,60,6)	(5,80,10)	(5,101,10)	(5,140,21)	(5,457,2)	(5,534,4)	(5,677,1)	(5,739,1)	(5,811,3)
(5,23,13)	(5,60,13)	(5,80,13)	(5,101,11)	(5,141,3)	(5,458,2)	(5,534,9)	(5,678,1)	(5,740,1)	(5,812,3)
(5,23,21)	(5,61,6)	(5,81,10)	(5,102,4)	(5,141,9)	(5,459,2)	(5,535,4)	(5,679,1)	(5,741,1)	(5,812,4)
(5,23,24)	(5,62,14)	(5,81,13)	(5,102,10)	(5,141,15)	(5,464,2)	(5,535,10)	(5,680,1)	(5,741,5)	(5,822,3)
(5,24,9)	(5,63,6)	(5,81,14)	(5,102,11)	(5,141,21)	(5,465,5)	(5,535,11)	(5,681,1)	(5,742,1)	(5,822,4)
(5,24,18)	(5,63,12)	(5,81,17)	(5,103,14)	(5,142,3)	(5,468,7)	(5,536,4)	(5,682,1)	(5,742,5)	(5,823,3)
(5,24,21)	(5,63,13)	(5,81,39)	(5,103,17)	(5,142,12)	(5,469,2)	(5,536,10)	(5,683,1)	(5,743,1)	(5,823,5)
(5,24,24)	(5,64,6)	(5,81,43)	(5,104,3)	(5,142,18)	(5,484,2)	(5,536,11)	(5,683,5)	(5,743,5)	(5,836,2)
(5,25,6)	(5,65,6)	(5,82,10)	(5,105,3)	(5,142,24)	(5,485,2)	(5,537,4)	(5,684,1)	(5,744,1)	(5,846,3)
(5,25,7)	(5,65,12)	(5,82,13)	(5,105,9)	(5,143,3)	(5,486,2)	(5,537,10)	(5,684,2)	(5,745,1)	(5,846,4)
(5,25,12)	(5,65,13)	(5,82,14)	(5,106,3)	(5,143,9)	(5,487,2)	(5,537,11)	(5,685,1)	(5,745,2)	(5,847,2)
(5,25,14)	(5,66,5)	(5,82,17)	(5,107,3)	(5,143,15)	(5,488,2)	(5,538,5)	(5,685,2)	(5,746,1)	(5,848,2)
(5,25,17)	(5,66,6)	(5,83,5)	(5,108,3)	(5,143,21)	(5,489,2)	(5,538,10)	(5,686,1)	(5,746,2)	(5,851,2)
(5,25,22)	(5,67,6)	(5,83,7)	(5,108,9)	(5,144,3)	(5,490,2)	(5,538,11)	(5,686,2)	(5,747,1)	(5,851,3)
(5,25,23)	(5,68,6)	(5,83,10)	(5,109,3)	(5,144,9)	(5,501,3)	(5,539,4)	(5,687,1)	(5,748,1)	(5,852,4)
(5,26,6)	(5,69,6)	(5,83,11)	(5,109,9)	(5,145,3)	(5,503,3)	(5,539,9)	(5,687,2)	(5,749,1)	(5,852,5)
(5,26,9)	(5,70,6)	(5,84,7)	(5,110,3)	(5,145,9)	(5,503,5)	(5,540,4)	(5,688,1)	(5,750,1)	(5,853,4)
(5,26,16)	(5,71,5)	(5,84,10)	(5,110,9)	(5,146,3)	(5,504,5)	(5,540,5)	(5,688,5)	(5,750,5)	(5,853,6)
(5,26,19)	(5,71,8)	(5,84,13)	(5,111,3)	(5,146,9)	(5,505,3)	(5,540,10)	(5,689,1)	(5,751,1)	(5,854,2)
(5,26,24)	(5,71,10)	(5,85,7)	(5,111,9)	(5,147,3)	(5,505,5)	(5,540,11)	(5,689,5)	(5,751,5)	(5,854,5)
(5,26,27)	(5,71,13)	(5,85,10)	(5,112,3)	(5,148,3)	(5,507,5)	(5,541,5)	(5,690,1)	(5,752,1)	(5,854,6)
(5,26,31)	(5,71,14)	(5,85,13)	(5,113,3)	(5,148,8)	(5,508,5)	(5,541,10)	(5,691,1)	(5,752,5)	(5,855,2)
(5,26,34)	(5,71,17)	(5,86,4)	(5,114,3)	(5,149,3)	(5,509,5)	(5,541,11)	(5,694,1)	(5,753,1)	(5,855,5)
(5,27,4)	(5,72,7)	(5,86,7)	(5,115,3)	(5,149,11)	(5,510,3)	(5,542,4)	(5,695,1)	(5,754,1)	(5,855,7)
(5,27,8)	(5,72,10)	(5,87,4)	(5,116,3)	(5,149,17)	(5,510,5)	(5,542,5)	(5,696,1)	(5,754,5)	(5,856,3)
(5,27,12)	(5,72,11)	(5,87,7)	(5,116,12)	(5,150,3)	(5,511,3)	(5,542,10)	(5,696,2)	(5,755,1)	(5,856,5)
(5,28,4)	(5,72,24)	(5,88,5)	(5,116,18)	(5,151,3)	(5,511,5)	(5,542,11)	(5,697,1)	(5,755,5)	(5,856,7)
(5,28,11)	(5,72,26)	(5,88,7)	(5,116,24)	(5,151,8)	(5,512,5)	(5,543,4)	(5,698,1)	(5,756,1)	(5,856,8)
(5,28,17)	(5,72,29)	(5,88,10)	(5,117,3)	(5,151,14)	(5,513,3)	(5,543,9)	(5,699,1)	(5,757,1)	(5,867,2)
(5,28,22)	(5,72,30)	(5,88,11)	(5,117,11)	(5,152,3)	(5,513,5)	(5,544,4)	(5,700,1)	(5,758,1)	(5,869,2)
(5,28,23)	(5,73,7)	(5,88,24)	(5,117,17)	(5,153,3)	(5,514,3)	(5,544,9)	(5,701,1)	(5,759,1)	(5,869,5)
(5,29,4)	(5,73,10)	(5,88,26)	(5,118,3)	(5,153,9)	(5,514,5)	(5,545,4)	(5,702,1)	(5,760,1)	(5,869,6)
(5,29,8)	(5,73,11)	(5,88,29)	(5,118,11)	(5,154,3)	(5,515,3)	(5,545,9)	(5,703,1)	(5,760,5)	(5,870,4)
(5,29,12)	(5,73,24)	(5,88,30)	(5,118,17)	(5,154,8)	(5,515,5)	(5,546,4)	(5,704,1)	(5,761,1)	(5,870,5)
(5,30,7)	(5,73,26)	(5,89,5)	(5,119,3)	(5,154,14)	(5,516,5)	(5,546,9)	(5,704,8)	(5,761,5)	(5,876,3)
(5,30,14)	(5,73,29)	(5,89,7)	(5,119,8)	(5,155,3)	(5,517,3)	(5,547,4)	(5,705,1)	(5,762,1)	(5,877,3)
(5,30,17)	(5,73,30)	(5,89,10)	(5,120,3)	(5,156,3)	(5,517,5)	(5,547,9)	(5,705,5)	(5,763,1)	(5,878,2)
(5,30,22)	(5,74,7)	(5,89,11)	(5,120,11)	(5,157,3)	(5,518,4)	(5,548,4)	(5,706,1)	(5,763,8)	(5,879,2)
(5,30,25)	(5,74,10)	(5,90,4)	(5,120,17)	(5,157,9)	(5,518,9)	(5,548,10)	(5,706,5)	(5,764,1)	(5,882,2)
(5,31,8)	(5,74,11)	(5,90,10)	(5,121,3)	(5,158,3)	(5,519,2)	(5,548,11)	(5,707,1)	(5,764,8)	(5,882,2)
(5,31,9)	(5,75,7)	(5,90,11)	(5,121,11)	(5,158,9)	(5,519,3)	(5,546,1)	(5,708,1)	(5,765,1)	(5,887,3)
(5,31,16)	(5,75,10)	(5,90,23)	(5,122,3)	(5,159,3)	(5,519,11)	(5,546,2)	(5,709,1)	(5,766,1)	(5,887,5)
(5,31,22)	(5,75,11)	(5,90,26)	(5,122,14)	(5,160,3)	(5,520,3)	(5,547,1)	(5,710,1)	(5,767,1)	(5,888,3)
(5,31,25)	(5,75,26)	(5,90,29)	(5,123,3)	(5,161,3)	(5,520,12)	(5,547,2)	(5,711,1)	(5,768,1)	(5,888,4)
(5,31,29)	(5,75,29)	(5,90,30)	(5,123,12)	(5,162,3)	(5,521,3)	(5,548,1)	(5,712,1)	(5,768,8)	(5,889,4)
(5,31,34)	(5,75,30)	(5,91,5)	(5,124,3)	(5,395,2)	(5,522,2)	(5,549,1)	(5,713,1)	(5,769,1)	(5,889,5)
(5,31,37)	(5,76,8)	(5,91,7)	(5,125,3)	(5,404,3)	(5,522,3)	(5,550,1)	(5,714,1)	(5,770,1)	(5,890,4)
(5,32,12)	(5,76,13)	(5,91,10)	(5,126,3)	(5,407,4)	(5,523,4)	(5,551,1)	(5,715,1)	(5,771,1)	(5,890,5)
(5,32,16)	(5,76,14)	(5,91,13)	(5,127,3)	(5,408,4)	(5,523,5)	(5,551,5)	(5,716,1)	(5,772,1)	(5,891,3)
(5,32,21)	(5,76,17)	(5,92,4)	(5,127,9)	(5,409,3)	(5,523,10)	(5,552,1)	(5,717,1)	(5,772,2)	(5,891,5)
(5,32,23)	(5,76,31)	(5,92,10)	(5,128,3)	(5,409,9)	(5,523,11)	(5,553,1)	(5,718,1)	(5,773,3)	(5,892,3)
(5,32,30)	(5,76,34)	(5,92,11)	(5,128,9)	(5,410,5)	(5,524,4)	(5,553,5)	(5,719,1)	(5,774,3)	(5,892,5)
(5,32,42)	(5,76,36)	(5,93,4)	(5,129,3)	(5,410,9)	(5,524,10)	(5,554,1)	(5,720,1)	(5,775,1)	(5,893,3)
(5,32,53)	(5,76,39)	(5,93,7)	(5,130,3)	(5,411,2)	(5,524,11)	(5,555,1)	(5,721,1)	(5,776,1)	(5,893,4)
(5,47,5)	(5,76,40)	(5,94,4)	(5,131,3)	(5,412,2)	(5,525,4)	(5,556,1)	(5,722,1)	(5,777,1)	(5,894,3)
(5,47,6)	(5,76,43)	(5,94,7)	(5,131,8)	(5,413,3)	(5,525,9)	(5,557,1)	(5,723,1)	(5,778,1)	(5,894,4)
(5,48,5)	(5,77,8)	(5,94,16)	(5,131,14)	(5,414,5)	(5,526,4)	(5,557,5)	(5,723,5)	(5,779,1)	(5,895,2)
(5,48,6)	(5,77,13)	(5,94,19)	(5,132,3)	(5,423,4)	(5,526,9)	(5,558,1)	(5,724,1)	(5,780,1)	(5,895,3)
(5,48,12)	(5,77,14)	(5,95,4)	(5,132,8)	(5,424,5)	(5,527,4)	(5,559,1)	(5,724,2)	(5,781,1)	(5,896,2)
(5,48,13)	(5,77,17)	(5,95,7)	(5,132,14)	(5,424,9)	(5,527,5)	(5,559,5)	(5,725,1)	(5,782,1)	(5,896,3)
(5,49,5)	(5,78,5)	(5,96,4)	(5,133,3)	(5,425,2)	(5,527,10)	(5,560,1)	(5,725,2)	(5,783,1)	
(5,49,6)	(5,78,7)	(5,96,7)	(5,133,11)	(5,426,3)	(5,527,11)	(5,561,1)	(5,726,1)	(5,784,1)	

Table 15:  $L(M)^G$  not retract  $k$ -rational, rank  $M = 5$ ,  $M$  : indecomposable (1141 cases)

CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT
(5,17,14)	(5,30,24)	(5,73,9)	(5,78,36)	(5,87,6)	(5,95,11)	(5,109,12)	(5,122,5)	(5,136,19)	(5,148,9)
(5,17,17)	(5,30,27)	(5,73,13)	(5,78,38)	(5,87,9)	(5,95,12)	(5,109,14)	(5,122,8)	(5,137,5)	(5,148,10)
(5,18,23)	(5,30,28)	(5,73,15)	(5,78,40)	(5,87,11)	(5,96,6)	(5,109,15)	(5,122,9)	(5,138,6)	(5,148,12)
(5,19,17)	(5,30,30)	(5,73,17)	(5,79,9)	(5,87,12)	(5,96,9)	(5,110,5)	(5,122,10)	(5,138,12)	(5,148,13)
(5,20,14)	(5,30,32)	(5,73,18)	(5,79,12)	(5,88,9)	(5,96,11)	(5,110,6)	(5,122,15)	(5,139,5)	(5,149,4)
(5,20,17)	(5,31,10)	(5,73,19)	(5,79,15)	(5,88,13)	(5,96,12)	(5,110,12)	(5,122,16)	(5,139,6)	(5,149,5)
(5,21,14)	(5,31,18)	(5,73,28)	(5,79,17)	(5,88,15)	(5,97,6)	(5,110,14)	(5,123,5)	(5,139,11)	(5,149,7)
(5,21,17)	(5,31,24)	(5,73,32)	(5,79,18)	(5,88,17)	(5,97,9)	(5,111,5)	(5,123,6)	(5,139,12)	(5,149,8)
(5,22,17)	(5,31,26)	(5,73,34)	(5,79,19)	(5,88,18)	(5,97,11)	(5,111,6)	(5,123,8)	(5,140,5)	(5,149,13)
(5,23,23)	(5,31,27)	(5,73,36)	(5,79,28)	(5,88,19)	(5,97,12)	(5,111,11)	(5,123,9)	(5,140,6)	(5,149,14)
(5,23,25)	(5,31,31)	(5,73,37)	(5,79,31)	(5,88,28)	(5,98,23)	(5,111,12)	(5,123,14)	(5,140,11)	(5,149,18)
(5,24,27)	(5,31,32)	(5,73,38)	(5,79,34)	(5,88,32)	(5,98,27)	(5,111,14)	(5,123,15)	(5,140,12)	(5,149,19)
(5,25,8)	(5,31,33)	(5,74,13)	(5,79,36)	(5,88,34)	(5,98,29)	(5,111,15)	(5,124,5)	(5,140,17)	(5,150,5)
(5,25,16)	(5,31,36)	(5,74,15)	(5,79,37)	(5,88,36)	(5,99,16)	(5,112,4)	(5,124,6)	(5,140,18)	(5,150,6)
(5,25,19)	(5,31,39)	(5,74,17)	(5,79,38)	(5,88,37)	(5,99,19)	(5,112,5)	(5,125,5)	(5,140,23)	(5,151,4)
(5,25,20)	(5,31,40)	(5,74,19)	(5,79,40)	(5,88,38)	(5,99,23)	(5,112,7)	(5,125,6)	(5,140,24)	(5,151,5)
(5,25,21)	(5,31,43)	(5,75,13)	(5,79,41)	(5,88,40)	(5,99,24)	(5,112,8)	(5,126,6)	(5,141,5)	(5,151,10)
(5,25,25)	(5,31,44)	(5,75,15)	(5,80,9)	(5,88,41)	(5,99,25)	(5,113,4)	(5,127,5)	(5,141,6)	(5,151,11)
(5,25,27)	(5,31,45)	(5,75,17)	(5,80,12)	(5,89,9)	(5,99,27)	(5,113,5)	(5,127,6)	(5,141,11)	(5,151,15)
(5,25,28)	(5,31,46)	(5,75,19)	(5,80,15)	(5,89,13)	(5,99,28)	(5,114,5)	(5,127,11)	(5,141,12)	(5,151,16)
(5,25,29)	(5,31,48)	(5,75,32)	(5,80,17)	(5,89,15)	(5,99,48)	(5,115,5)	(5,127,12)	(5,141,17)	(5,151,18)
(5,25,30)	(5,32,18)	(5,75,34)	(5,80,18)	(5,89,17)	(5,99,55)	(5,115,6)	(5,128,5)	(5,141,18)	(5,151,19)
(5,25,32)	(5,32,25)	(5,75,36)	(5,80,19)	(5,89,18)	(5,99,58)	(5,116,5)	(5,128,6)	(5,141,23)	(5,152,4)
(5,25,33)	(5,32,34)	(5,75,38)	(5,81,16)	(5,89,19)	(5,100,12)	(5,116,6)	(5,128,11)	(5,141,24)	(5,152,5)
(5,26,10)	(5,32,35)	(5,75,40)	(5,81,19)	(5,90,9)	(5,100,16)	(5,116,8)	(5,128,12)	(5,141,26)	(5,153,5)
(5,26,18)	(5,32,36)	(5,76,12)	(5,81,22)	(5,90,13)	(5,100,19)	(5,116,9)	(5,129,5)	(5,141,27)	(5,153,6)
(5,26,21)	(5,32,41)	(5,76,16)	(5,81,23)	(5,90,16)	(5,100,23)	(5,116,14)	(5,129,6)	(5,142,5)	(5,153,11)
(5,26,26)	(5,32,44)	(5,76,19)	(5,81,24)	(5,90,17)	(5,100,24)	(5,116,15)	(5,130,5)	(5,142,6)	(5,153,12)
(5,26,28)	(5,32,51)	(5,76,22)	(5,81,26)	(5,90,18)	(5,100,25)	(5,116,20)	(5,130,6)	(5,142,8)	(5,154,4)
(5,26,29)	(5,32,54)	(5,76,23)	(5,81,45)	(5,90,19)	(5,100,27)	(5,116,21)	(5,131,4)	(5,142,9)	(5,154,5)
(5,26,30)	(5,32,57)	(5,76,24)	(5,81,52)	(5,90,28)	(5,100,28)	(5,116,26)	(5,131,5)	(5,142,14)	(5,154,10)
(5,26,33)	(5,32,60)	(5,76,25)	(5,81,55)	(5,90,32)	(5,101,13)	(5,116,27)	(5,131,10)	(5,142,15)	(5,154,11)
(5,26,36)	(5,36,3)	(5,76,26)	(5,82,16)	(5,90,35)	(5,101,17)	(5,117,5)	(5,131,11)	(5,142,20)	(5,154,15)
(5,26,38)	(5,36,7)	(5,76,38)	(5,82,19)	(5,90,36)	(5,101,18)	(5,117,7)	(5,131,16)	(5,142,21)	(5,154,16)
(5,26,39)	(5,38,6)	(5,76,42)	(5,82,22)	(5,90,37)	(5,101,20)	(5,117,14)	(5,131,18)	(5,142,26)	(5,155,4)
(5,26,40)	(5,39,5)	(5,76,45)	(5,82,23)	(5,90,38)	(5,101,21)	(5,117,19)	(5,132,4)	(5,142,27)	(5,155,5)
(5,26,41)	(5,71,12)	(5,76,48)	(5,82,24)	(5,91,9)	(5,102,9)	(5,118,5)	(5,132,5)	(5,143,5)	(5,156,5)
(5,26,42)	(5,71,16)	(5,76,49)	(5,82,26)	(5,91,12)	(5,102,13)	(5,118,7)	(5,132,10)	(5,143,6)	(5,157,5)
(5,27,5)	(5,71,19)	(5,76,50)	(5,83,9)	(5,91,15)	(5,102,17)	(5,118,13)	(5,132,11)	(5,143,11)	(5,157,6)
(5,27,10)	(5,71,22)	(5,76,51)	(5,83,13)	(5,91,17)	(5,102,18)	(5,118,14)	(5,132,15)	(5,143,12)	(5,157,11)
(5,27,11)	(5,71,23)	(5,76,53)	(5,83,15)	(5,91,18)	(5,102,20)	(5,118,18)	(5,132,16)	(5,143,17)	(5,157,12)
(5,27,14)	(5,71,24)	(5,76,54)	(5,83,17)	(5,91,19)	(5,102,21)	(5,118,19)	(5,132,18)	(5,143,18)	(5,158,5)
(5,27,15)	(5,71,25)	(5,76,55)	(5,83,18)	(5,92,13)	(5,103,23)	(5,119,4)	(5,132,19)	(5,143,23)	(5,158,6)
(5,27,16)	(5,71,26)	(5,77,12)	(5,83,19)	(5,92,17)	(5,103,27)	(5,119,5)	(5,133,5)	(5,143,24)	(5,158,11)
(5,28,19)	(5,71,28)	(5,77,16)	(5,83,21)	(5,92,18)	(5,103,29)	(5,119,9)	(5,133,7)	(5,144,5)	(5,158,12)
(5,28,20)	(5,71,29)	(5,77,19)	(5,83,22)	(5,92,19)	(5,104,4)	(5,119,10)	(5,133,14)	(5,144,6)	(5,159,5)
(5,28,21)	(5,72,9)	(5,77,22)	(5,84,12)	(5,93,6)	(5,104,5)	(5,119,12)	(5,133,19)	(5,144,11)	(5,159,6)
(5,28,25)	(5,72,13)	(5,77,23)	(5,84,15)	(5,93,9)	(5,105,5)	(5,119,13)	(5,134,4)	(5,144,12)	(5,160,4)
(5,28,28)	(5,72,15)	(5,77,24)	(5,84,17)	(5,93,11)	(5,105,6)	(5,120,4)	(5,134,5)	(5,145,5)	(5,160,5)
(5,28,29)	(5,72,17)	(5,77,25)	(5,84,19)	(5,93,12)	(5,105,11)	(5,120,5)	(5,134,9)	(5,145,6)	(5,160,7)
(5,28,30)	(5,72,18)	(5,77,26)	(5,85,9)	(5,94,6)	(5,105,12)	(5,120,7)	(5,134,10)	(5,145,11)	(5,161,5)
(5,29,5)	(5,72,19)	(5,78,9)	(5,85,12)	(5,94,9)	(5,106,6)	(5,120,8)	(5,135,4)	(5,145,12)	(5,161,7)
(5,29,10)	(5,72,28)	(5,78,12)	(5,85,15)	(5,94,11)	(5,107,5)	(5,120,13)	(5,135,5)	(5,146,5)	(5,162,5)
(5,29,11)	(5,72,32)	(5,78,15)	(5,85,17)	(5,94,12)	(5,107,6)	(5,120,14)	(5,136,4)	(5,146,6)	(5,162,7)
(5,29,14)	(5,72,34)	(5,78,17)	(5,85,18)	(5,94,18)	(5,108,6)	(5,120,19)	(5,136,5)	(5,146,11)	(5,224,9)
(5,29,15)	(5,72,36)	(5,78,18)	(5,85,19)	(5,94,21)	(5,108,12)	(5,120,21)	(5,136,7)	(5,146,12)	(5,227,11)
(5,29,16)	(5,72,37)	(5,78,19)	(5,86,6)	(5,94,23)	(5,108,14)	(5,121,5)	(5,136,8)	(5,147,4)	(5,232,14)
(5,30,19)	(5,72,38)	(5,78,28)	(5,86,9)	(5,94,24)	(5,109,5)	(5,121,7)	(5,136,13)	(5,147,5)	(5,233,9)
(5,30,20)	(5,72,40)	(5,78,31)	(5,86,11)	(5,95,6)	(5,109,6)	(5,121,13)	(5,136,14)	(5,148,4)	(5,233,10)
(5,30,21)	(5,72,41)	(5,78,34)	(5,86,12)	(5,95,9)	(5,109,11)	(5,122,4)	(5,136,18)	(5,148,5)	(5,234,11)

Table 15 (continued):  $L(M)^G$  not retract  $k$ -rational, rank  $M = 5$ ,  $M$  : indecomposable (1141 cases)

CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT	CARAT
(5,234,12)	(5,519,15)	(5,533,13)	(5,547,11)	(5,568,10)	(5,639,2)	(5,685,5)	(5,723,2)	(5,764,3)	(5,939,2)
(5,240,9)	(5,520,7)	(5,533,14)	(5,547,12)	(5,569,2)	(5,641,4)	(5,686,3)	(5,723,4)	(5,764,5)	(5,939,3)
(5,240,10)	(5,520,9)	(5,534,6)	(5,548,7)	(5,569,4)	(5,642,2)	(5,686,5)	(5,724,3)	(5,764,7)	(5,940,2)
(5,241,14)	(5,520,15)	(5,534,7)	(5,548,8)	(5,569,7)	(5,643,2)	(5,687,3)	(5,724,5)	(5,764,9)	(5,940,3)
(5,241,15)	(5,521,7)	(5,534,11)	(5,548,13)	(5,569,10)	(5,651,2)	(5,688,2)	(5,725,3)	(5,764,11)	(5,941,2)
(5,242,9)	(5,521,9)	(5,534,12)	(5,548,14)	(5,570,2)	(5,651,4)	(5,688,4)	(5,725,5)	(5,765,2)	(5,941,3)
(5,244,9)	(5,522,5)	(5,535,7)	(5,549,2)	(5,570,4)	(5,651,6)	(5,689,2)	(5,726,2)	(5,765,4)	(5,942,2)
(5,244,10)	(5,522,7)	(5,535,8)	(5,549,5)	(5,570,7)	(5,651,8)	(5,696,3)	(5,727,2)	(5,766,2)	(5,942,3)
(5,246,11)	(5,522,8)	(5,535,13)	(5,550,2)	(5,570,10)	(5,652,2)	(5,697,2)	(5,727,4)	(5,766,4)	(5,943,2)
(5,246,12)	(5,523,7)	(5,535,14)	(5,550,4)	(5,571,2)	(5,652,4)	(5,697,4)	(5,728,3)	(5,767,2)	(5,943,3)
(5,257,14)	(5,523,8)	(5,536,7)	(5,550,7)	(5,571,4)	(5,653,2)	(5,700,2)	(5,728,5)	(5,767,4)	(5,944,2)
(5,257,15)	(5,523,13)	(5,536,8)	(5,550,10)	(5,571,7)	(5,653,4)	(5,701,2)	(5,729,3)	(5,768,2)	(5,944,3)
(5,274,4)	(5,523,14)	(5,536,13)	(5,551,2)	(5,571,10)	(5,653,6)	(5,701,4)	(5,729,5)	(5,768,3)	(5,945,2)
(5,275,4)	(5,523,16)	(5,536,14)	(5,551,4)	(5,572,2)	(5,653,8)	(5,702,2)	(5,741,2)	(5,768,5)	(5,945,3)
(5,276,4)	(5,523,17)	(5,537,8)	(5,551,7)	(5,572,5)	(5,654,2)	(5,702,4)	(5,741,4)	(5,768,7)	(5,946,1)
(5,290,6)	(5,524,8)	(5,537,13)	(5,551,10)	(5,573,2)	(5,654,4)	(5,703,2)	(5,742,2)	(5,768,9)	(5,946,3)
(5,291,6)	(5,524,13)	(5,537,14)	(5,552,2)	(5,573,5)	(5,655,2)	(5,704,2)	(5,743,2)	(5,768,11)	(5,947,1)
(5,294,6)	(5,524,14)	(5,538,8)	(5,552,4)	(5,574,2)	(5,656,2)	(5,704,3)	(5,743,4)	(5,769,2)	(5,947,3)
(5,302,4)	(5,525,6)	(5,538,13)	(5,552,7)	(5,574,4)	(5,657,2)	(5,704,5)	(5,749,2)	(5,769,4)	(5,948,1)
(5,303,4)	(5,525,7)	(5,538,14)	(5,552,10)	(5,574,7)	(5,657,4)	(5,704,9)	(5,749,4)	(5,770,2)	(5,948,2)
(5,304,4)	(5,525,11)	(5,539,6)	(5,553,2)	(5,574,10)	(5,657,6)	(5,705,2)	(5,750,2)	(5,770,4)	(5,948,3)
(5,305,4)	(5,525,12)	(5,539,7)	(5,553,5)	(5,575,2)	(5,657,8)	(5,705,6)	(5,750,6)	(5,772,3)	(5,948,4)
(5,305,8)	(5,526,6)	(5,539,11)	(5,554,2)	(5,575,5)	(5,658,2)	(5,705,7)	(5,751,2)	(5,772,5)	(5,949,1)
(5,306,12)	(5,526,7)	(5,539,12)	(5,554,5)	(5,576,2)	(5,658,4)	(5,706,2)	(5,751,4)	(5,773,4)	(5,949,2)
(5,316,4)	(5,526,11)	(5,540,7)	(5,555,2)	(5,576,5)	(5,659,2)	(5,706,6)	(5,751,6)	(5,774,4)	(5,949,3)
(5,326,6)	(5,526,12)	(5,540,8)	(5,555,5)	(5,577,2)	(5,659,4)	(5,706,7)	(5,751,8)	(5,785,3)	(5,949,4)
(5,335,6)	(5,527,7)	(5,540,13)	(5,556,2)	(5,577,5)	(5,659,6)	(5,706,8)	(5,752,2)	(5,786,3)	(5,950,1)
(5,336,8)	(5,527,8)	(5,540,14)	(5,556,5)	(5,578,2)	(5,659,8)	(5,706,10)	(5,752,4)	(5,786,5)	(5,950,2)
(5,336,12)	(5,527,13)	(5,540,16)	(5,557,2)	(5,578,5)	(5,660,2)	(5,706,12)	(5,752,6)	(5,801,6)	(5,950,3)
(5,337,8)	(5,527,14)	(5,540,17)	(5,557,5)	(5,579,2)	(5,660,4)	(5,707,2)	(5,752,8)	(5,822,5)	(5,950,4)
(5,338,6)	(5,527,16)	(5,541,8)	(5,558,2)	(5,579,5)	(5,664,2)	(5,708,2)	(5,753,2)	(5,823,4)	(5,951,1)
(5,339,6)	(5,527,17)	(5,541,13)	(5,558,5)	(5,580,2)	(5,664,4)	(5,708,4)	(5,754,2)	(5,846,5)	(5,951,2)
(5,340,8)	(5,528,7)	(5,541,14)	(5,559,2)	(5,604,2)	(5,665,4)	(5,709,2)	(5,754,6)	(5,852,6)	(5,951,3)
(5,340,12)	(5,528,8)	(5,542,7)	(5,559,5)	(5,604,4)	(5,666,4)	(5,709,3)	(5,755,2)	(5,853,5)	(5,952,1)
(5,342,4)	(5,528,13)	(5,542,8)	(5,560,2)	(5,605,2)	(5,667,2)	(5,709,5)	(5,755,4)	(5,854,7)	(5,952,3)
(5,372,6)	(5,528,15)	(5,542,13)	(5,560,5)	(5,605,4)	(5,667,4)	(5,710,2)	(5,755,6)	(5,855,6)	(5,953,1)
(5,404,5)	(5,528,16)	(5,542,14)	(5,561,2)	(5,606,2)	(5,668,2)	(5,711,2)	(5,755,8)	(5,856,4)	(5,953,2)
(5,409,5)	(5,528,17)	(5,543,6)	(5,561,5)	(5,607,2)	(5,669,2)	(5,712,2)	(5,756,2)	(5,869,7)	(5,953,3)
(5,410,7)	(5,529,8)	(5,543,7)	(5,562,2)	(5,607,6)	(5,669,4)	(5,712,4)	(5,757,2)	(5,870,6)	(5,954,1)
(5,413,6)	(5,529,13)	(5,543,11)	(5,562,5)	(5,608,2)	(5,670,2)	(5,713,2)	(5,757,4)	(5,889,6)	(5,954,2)
(5,414,8)	(5,529,15)	(5,543,12)	(5,563,2)	(5,608,6)	(5,670,3)	(5,714,2)	(5,758,2)	(5,890,6)	(5,954,3)
(5,424,7)	(5,529,16)	(5,544,6)	(5,563,5)	(5,621,2)	(5,670,7)	(5,714,4)	(5,759,2)	(5,891,4)	(5,954,4)
(5,426,6)	(5,530,7)	(5,544,7)	(5,564,2)	(5,621,4)	(5,671,2)	(5,715,2)	(5,759,4)	(5,892,4)	(5,955,1)
(5,434,5)	(5,530,8)	(5,544,11)	(5,564,5)	(5,622,2)	(5,671,3)	(5,716,2)	(5,760,2)	(5,933,2)	(5,955,2)
(5,435,6)	(5,530,13)	(5,544,12)	(5,565,2)	(5,622,4)	(5,671,7)	(5,716,4)	(5,760,6)	(5,933,3)	(5,955,3)
(5,436,5)	(5,530,15)	(5,545,6)	(5,565,4)	(5,623,2)	(5,672,2)	(5,717,2)	(5,761,2)	(5,934,2)	(5,955,4)
(5,465,7)	(5,530,16)	(5,545,7)	(5,565,7)	(5,623,6)	(5,673,2)	(5,717,4)	(5,761,4)	(5,934,3)	
(5,501,5)	(5,530,17)	(5,545,11)	(5,565,10)	(5,624,2)	(5,673,4)	(5,718,2)	(5,761,6)	(5,935,2)	
(5,518,6)	(5,531,8)	(5,545,12)	(5,566,2)	(5,624,4)	(5,674,2)	(5,718,4)	(5,761,8)	(5,935,3)	
(5,518,7)	(5,531,15)	(5,546,6)	(5,566,5)	(5,624,6)	(5,675,2)	(5,719,2)	(5,762,2)	(5,936,2)	
(5,518,11)	(5,531,17)	(5,546,7)	(5,567,2)	(5,624,9)	(5,675,4)	(5,719,4)	(5,763,2)	(5,936,3)	
(5,518,12)	(5,532,6)	(5,546,11)	(5,567,5)	(5,625,2)	(5,683,2)	(5,720,2)	(5,763,3)	(5,937,2)	
(5,519,5)	(5,532,7)	(5,546,12)	(5,568,2)	(5,625,4)	(5,683,4)	(5,720,4)	(5,763,5)	(5,937,3)	
(5,519,7)	(5,532,11)	(5,547,6)	(5,568,4)	(5,638,2)	(5,684,3)	(5,721,2)	(5,763,9)	(5,938,2)	
(5,519,8)	(5,532,12)	(5,547,7)	(5,568,7)	(5,638,4)	(5,685,3)	(5,721,4)	(5,764,2)	(5,938,3)	

Table 16: birational classification of the algebraic  $k$ -tori of dimension 5

CARAT	#	[s, r, u]	$G(n, i)$	CARAT	#	[s, r, u]	$G(n, i)$		
(5,1)	1	[1,0,0]	$G(1, 1)$	{1}	(5,81)	55	[34,0,21]	$G(16, 11)$	$C_2 \times D_4$
(5,2)	1	[1,0,0]	$G(2, 1)$	$C_2$	(5,82)	26	[12,0,14]	$G(16, 11)$	$C_2 \times D_4$
(5,3)	2	[2,0,0]	$G(2, 1)$	$C_2$	(5,83)	22	[4,0,18]	$G(16, 10)$	$C_4 \times C_2^2$
(5,4)	2	[2,0,0]	$G(2, 1)$	$C_2$	(5,84)	19	[10,0,9]	$G(16, 10)$	$C_4 \times C_2^2$
(5,5)	2	[2,0,0]	$G(4, 2)$	$C_2^2$	(5,85)	19	[6,0,13]	$G(16, 10)$	$C_4 \times C_2^2$
(5,6)	3	[3,0,0]	$G(2, 1)$	$C_2$	(5,86)	12	[3,0,9]	$G(16, 10)$	$C_4 \times C_2^2$
(5,7)	3	[3,0,0]	$G(2, 1)$	$C_2$	(5,87)	12	[2,0,10]	$G(16, 10)$	$C_4 \times C_2^2$
(5,8)	3	[3,0,0]	$G(4, 2)$	$C_2^2$	(5,88)	41	[7,0,34]	$G(32, 46)$	$C_2^2 \times D_4$
(5,9)	6	[6,0,0]	$G(4, 2)$	$C_2^2$	(5,89)	19	[3,0,16]	$G(32, 46)$	$C_2^2 \times D_4$
(5,10)	7	[7,0,0]	$G(4, 2)$	$C_2^2$	(5,90)	38	[9,0,29]	$G(32, 46)$	$C_2^2 \times D_4$
(5,11)	6	[5,0,1]	$G(4, 2)$	$C_2^2$	(5,91)	19	[3,0,16]	$G(32, 46)$	$C_2^2 \times D_4$
(5,12)	6	[4,0,2]	$G(8, 5)$	$C_2^2$	(5,92)	19	[10,0,9]	$G(32, 46)$	$C_2^2 \times D_4$
(5,13)	9	[9,0,0]	$G(4, 2)$	$C_2^2$	(5,93)	12	[3,0,9]	$G(32, 46)$	$C_2^2 \times D_4$
(5,14)	9	[7,0,2]	$G(4, 2)$	$C_2^2$	(5,94)	24	[4,0,20]	$G(32, 46)$	$C_2^2 \times D_4$
(5,15)	11	[11,0,0]	$G(4, 2)$	$C_2^2$	(5,95)	12	[2,0,10]	$G(32, 46)$	$C_2^2 \times D_4$
(5,16)	9	[5,0,4]	$G(8, 5)$	$C_2^2$	(5,96)	12	[2,0,10]	$G(32, 45)$	$C_4 \times C_2^3$
(5,17)	17	[5,0,12]	$G(16, 14)$	$C_2^4$	(5,97)	12	[2,0,10]	$G(64, 261)$	$C_2^3 \times D_4$
(5,18)	28	[23,0,5]	$G(4, 2)$	$C_2^2$	(5,98)	29	[22,0,7]	$G(8, 3)$	$D_4$
(5,19)	18	[15,0,3]	$G(4, 2)$	$C_2^2$	(5,99)	58	[36,0,22]	$G(8, 3)$	$D_4$
(5,20)	18	[6,0,12]	$G(8, 5)$	$C_2^2$	(5,100)	29	[12,0,17]	$G(8, 3)$	$D_4$
(5,21)	17	[7,0,10]	$G(8, 5)$	$C_2^2$	(5,101)	22	[12,0,10]	$G(8, 2)$	$C_4 \times C_2$
(5,22)	17	[12,0,5]	$G(8, 5)$	$C_2^2$	(5,102)	22	[10,0,12]	$G(8, 2)$	$C_4 \times C_2$
(5,23)	27	[19,0,8]	$G(8, 5)$	$C_2^2$	(5,103)	29	[22,0,7]	$G(8, 2)$	$C_4 \times C_2$
(5,24)	27	[13,0,14]	$G(8, 5)$	$C_2^2$	(5,104)	5	[1,0,4]	$G(128, 2194)$	$C_2 \times D_4 \times D_4$
(5,25)	33	[6,0,27]	$G(16, 14)$	$C_2^4$	(5,105)	12	[3,0,9]	$G(16, 4)$	$C_4 \times C_4$
(5,26)	42	[10,0,32]	$G(16, 14)$	$C_2^4$	(5,106)	6	[4,0,2]	$G(16, 4)$	$C_4 \times C_4$
(5,27)	16	[3,0,13]	$G(16, 14)$	$C_2^4$	(5,107)	6	[2,0,4]	$G(16, 4)$	$C_4 \times C_4$
(5,28)	30	[14,0,16]	$G(16, 14)$	$C_2^4$	(5,108)	15	[10,0,5]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,29)	16	[3,0,13]	$G(32, 51)$	$C_2^5$	(5,109)	15	[3,0,12]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,30)	33	[16,0,17]	$G(8, 5)$	$C_2^3$	(5,110)	15	[8,0,7]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,31)	49	[15,0,34]	$G(8, 5)$	$C_2^3$	(5,111)	15	[4,0,11]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,32)	60	[36,0,24]	$G(8, 5)$	$C_2^3$	(5,112)	8	[2,0,6]	$G(16, 10)$	$C_4 \times C_2^2$
(5,33)	2	[2,0,0]	$G(4, 1)$	$C_4$	(5,113)	5	[1,0,4]	$G(16, 2)$	$C_4 \times C_4$
(5,34)	2	[2,0,0]	$G(4, 1)$	$C_4$	(5,114)	5	[3,0,2]	$G(16, 2)$	$C_4 \times C_4$
(5,35)	2	[1,0,1]	$G(8, 2)$	$C_4 \times C_2$	(5,115)	6	[2,0,4]	$G(16, 2)$	$C_4 \times C_4$
(5,36)	7	[3,0,4]	$G(16, 11)$	$C_2 \times D_4$	(5,116)	27	[7,0,20]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,37)	7	[7,0,0]	$G(8, 3)$	$D_4$	(5,117)	19	[12,0,7]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,38)	10	[7,0,3]	$G(8, 3)$	$D_4$	(5,118)	19	[8,0,11]	$G(16, 3)$	$(C_4 \times C_2) \rtimes C_2$
(5,39)	7	[4,0,3]	$G(8, 3)$	$D_4$	(5,119)	13	[3,0,10]	$G(16, 11)$	$C_2 \times D_4$
(5,40)	2	[1,0,1]	$G(16, 11)$	$C_2 \times D_4$	(5,120)	22	[9,0,13]	$G(16, 11)$	$C_2 \times D_4$
(5,41)	2	[2,0,0]	$G(4, 1)$	$C_4$	(5,121)	13	[8,0,5]	$G(16, 11)$	$C_2 \times D_4$
(5,42)	2	[2,0,0]	$G(4, 1)$	$C_4$	(5,122)	16	[5,0,11]	$G(16, 11)$	$C_2 \times D_4$
(5,43)	2	[2,0,0]	$G(8, 3)$	$D_4$	(5,123)	15	[3,0,12]	$G(32, 22)$	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$
(5,44)	2	[1,0,1]	$G(8, 3)$	$D_4$	(5,124)	6	[1,0,5]	$G(32, 23)$	$C_2 \times (C_4 \rtimes C_4)$
(5,45)	4	[3,0,1]	$G(8, 3)$	$D_4$	(5,125)	6	[1,0,5]	$G(32, 25)$	$C_4 \times D_4$
(5,46)	2	[1,0,1]	$G(8, 2)$	$C_4 \times C_2$	(5,126)	6	[4,0,2]	$G(32, 25)$	$C_4 \times D_4$
(5,47)	7	[2,0,5]	$G(16, 11)$	$C_2 \times D_4$	(5,127)	12	[2,0,10]	$G(32, 25)$	$C_4 \times D_4$
(5,48)	15	[5,0,10]	$G(16, 11)$	$C_2 \times D_4$	(5,128)	12	[3,0,9]	$G(32, 25)$	$C_4 \times D_4$
(5,49)	14	[6,0,8]	$G(16, 11)$	$C_2 \times D_4$	(5,129)	6	[2,0,4]	$G(32, 25)$	$C_4 \times D_4$
(5,50)	7	[2,0,5]	$G(16, 11)$	$C_2 \times D_4$	(5,130)	6	[1,0,5]	$G(32, 25)$	$C_4 \times D_4$
(5,51)	7	[4,0,3]	$G(16, 11)$	$C_2 \times D_4$	(5,131)	19	[8,0,11]	$G(32, 27)$	$C_2^4 \times C_2$
(5,52)	14	[7,0,7]	$G(16, 11)$	$C_2 \times D_4$	(5,132)	19	[4,0,15]	$G(32, 27)$	$C_2^4 \times C_2$
(5,53)	7	[2,0,5]	$G(16, 11)$	$C_2 \times D_4$	(5,133)	19	[12,0,7]	$G(32, 27)$	$C_2^4 \times C_2$
(5,54)	7	[5,0,2]	$G(16, 11)$	$C_2 \times D_4$	(5,134)	10	[2,0,8]	$G(32, 34)$	$C_2^2 \times C_2$
(5,55)	7	[2,0,5]	$G(16, 10)$	$C_4 \times C_2^2$	(5,135)	5	[1,0,4]	$G(32, 34)$	$C_2^4 \times C_2$
(5,56)	7	[2,0,5]	$G(32, 46)$	$C_2^2 \times D_4$	(5,136)	19	[4,0,15]	$G(32, 27)$	$C_2^4 \times C_2$
(5,57)	8	[8,0,0]	$G(4, 1)$	$C_4$	(5,137)	5	[3,0,2]	$G(32, 34)$	$C_2^4 \times C_2$
(5,58)	8	[8,0,0]	$G(4, 1)$	$C_4$	(5,138)	12	[8,0,4]	$G(32, 28)$	$(C_4 \times C_2^2) \rtimes C_2$
(5,59)	7	[4,0,3]	$G(8, 3)$	$D_4$	(5,139)	12	[3,0,9]	$G(32, 28)$	$(C_4 \times C_2^2) \rtimes C_2$
(5,60)	14	[9,0,5]	$G(8, 3)$	$D_4$	(5,140)	24	[6,0,18]	$G(32, 28)$	$(C_4 \times C_2^2) \rtimes C_2$
(5,61)	7	[5,0,2]	$G(8, 3)$	$D_4$	(5,141)	27	[7,0,20]	$G(32, 27)$	$C_2^4 \times C_2$
(5,62)	15	[13,0,2]	$G(8, 3)$	$D_4$	(5,142)	27	[5,0,22]	$G(32, 27)$	$C_2^4 \times C_2$
(5,63)	15	[8,0,7]	$G(8, 3)$	$D_4$	(5,143)	24	[4,0,20]	$G(32, 28)$	$(C_4 \times C_2^2) \rtimes C_2$
(5,64)	15	[12,0,3]	$G(8, 3)$	$D_4$	(5,144)	12	[3,0,9]	$G(32, 28)$	$(C_4 \times C_2^2) \rtimes C_2$
(5,65)	15	[7,0,8]	$G(8, 3)$	$D_4$	(5,145)	12	[2,0,10]	$G(32, 28)$	$(C_4 \times C_2^2) \rtimes C_2$
(5,66)	8	[3,0,5]	$G(8, 2)$	$C_4 \times C_2$	(5,146)	12	[3,0,9]	$G(32, 34)$	$C_2^4 \times C_2$
(5,67)	7	[5,0,2]	$G(8, 2)$	$C_4 \times C_2$	(5,147)	5	[1,0,4]	$G(32, 21)$	$C_4 \times C_4 \times C_2$
(5,68)	7	[5,0,2]	$G(8, 2)$	$C_4 \times C_2$	(5,148)	13	[3,0,10]	$G(32, 46)$	$C_2^2 \times D_4$
(5,69)	7	[4,0,3]	$G(8, 2)$	$C_4 \times C_2$	(5,149)	19	[4,0,15]	$G(32, 22)$	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$
(5,70)	7	[4,0,3]	$G(8, 2)$	$C_4 \times C_2$	(5,150)	6	[1,0,5]	$G(64, 196)$	$C_2 \times C_4 \times D_4$
(5,71)	29	[5,0,24]	$G(16, 11)$	$C_2 \times D_4$	(5,151)	19	[4,0,15]	$G(64, 202)$	$C_2 \times (C_2^4 \rtimes C_2)$
(5,72)	41	[10,0,31]	$G(16, 11)$	$C_2 \times D_4$	(5,152)	5	[1,0,4]	$G(64, 211)$	$C_2 \times (C_2^4 \rtimes C_2)$
(5,73)	38	[9,0,29]	$G(16, 11)$	$C_2 \times D_4$	(5,153)	12	[2,0,10]	$G(64, 203)$	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$
(5,74)	19	[10,0,9]	$G(16, 11)$	$C_2 \times D_4$	(5,154)	16	[3,0,13]	$G(64, 226)$	$D_4 \times D_4$
(5,75)	41	[22,0,19]	$G(16, 11)$	$C_2 \times D_4$	(5,155)	5	[1,0,4]	$G(64, 226)$	$D_4 \times D_4$
(5,76)	55	[13,0,42]	$G(16, 11)$	$C_2 \times D_4$	(5,156)	5	[3,0,2]	$G(64, 226)$	$D_4 \times D_4$
(5,77)	26	[8,0,18]	$G(16, 11)$	$C_2 \times D_4$	(5,157)	12	[3,0,9]	$G(64, 226)$	$D_4 \times D_4$
(5,78)	41	[14,0,27]	$G(16, 11)$	$C_2 \times D_4$	(5,158)	12	[2,0,10]	$G(64, 226)$	$D_4 \times D_4$
(5,79)	41	[7,0,34]	$G(16, 11)$	$C_2 \times D_4$	(5,159)	6	[1,0,5]	$G(64, 226)$	$D_4 \times D_4$
(5,80)	19	[4,0,15]	$G(16, 11)$	$C_2 \times D_4$	(5,160)	8	[3,0,5]	$G(8, 2)$	$C_4 \times C_2$

Table 16 (continued): birational classification of the algebraic  $k$ -tori of dimension 5

CARAT	#	$[s, r, u]$	$G(n, i)$	CARAT	#	$[s, r, u]$	$G(n, i)$		
(5,161)	8	[5,0,3]	$G(8, 2)$	$C_4 \times C_2$	(5,241)	15	[9,0,6]	$G(24, 14)$	$C_2^2 \times S_3$
(5,162)	8	[5,0,3]	$G(8, 2)$	$C_4 \times C_2$	(5,242)	15	[12,0,3]	$G(24, 14)$	$C_2^2 \times S_3$
(5,163)	1	[1,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,243)	15	[15,0,0]	$G(24, 14)$	$C_2^2 \times S_3$
(5,164)	2	[2,0,0]	$G(3, 1)$	$C_3$	(5,244)	15	[9,0,6]	$G(24, 14)$	$C_2^2 \times S_3$
(5,165)	1	[1,0,0]	$G(6, 2)$	$C_6$	(5,245)	18	[18,0,0]	$G(24, 14)$	$C_2^2 \times S_3$
(5,166)	1	[1,0,0]	$G(6, 2)$	$C_6$	(5,246)	18	[12,0,6]	$G(24, 14)$	$C_2^2 \times S_3$
(5,167)	2	[2,0,0]	$G(6, 2)$	$C_6$	(5,247)	8	[6,0,2]	$G(24, 14)$	$C_2^2 \times S_3$
(5,168)	2	[2,0,0]	$G(12, 4)$	$D_{12}$	(5,248)	8	[4,0,4]	$G(24, 14)$	$C_2 \times S_3$
(5,169)	2	[2,0,0]	$G(12, 4)$	$D_{12}$	(5,249)	10	[6,0,4]	$G(24, 14)$	$C_2^2 \times S_3$
(5,170)	3	[3,0,0]	$G(12, 4)$	$D_{12}$	(5,250)	10	[10,0,0]	$G(24, 14)$	$C_2 \times S_3$
(5,171)	5	[5,0,0]	$G(12, 4)$	$D_{12}$	(5,251)	10	[6,0,4]	$G(24, 14)$	$C_2 \times S_3$
(5,172)	2	[2,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,252)	5	[5,0,0]	$G(24, 14)$	$C_2 \times S_3$
(5,173)	5	[5,0,0]	$G(6, 1)$	$S_3$	(5,253)	5	[4,0,1]	$G(24, 14)$	$C_2 \times S_3$
(5,174)	5	[5,0,0]	$G(6, 1)$	$S_3$	(5,254)	6	[4,0,2]	$G(24, 14)$	$C_2^2 \times S_3$
(5,175)	1	[1,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,255)	6	[6,0,0]	$G(24, 14)$	$C_2^2 \times S_3$
(5,176)	3	[3,0,0]	$G(12, 4)$	$D_{12}$	(5,256)	4	[2,0,2]	$G(48, 52)$	$C_6 \times C_2^3$
(5,177)	2	[2,0,0]	$G(12, 4)$	$D_{12}$	(5,257)	15	[9,0,6]	$G(48, 51)$	$C_2^3 \times S_3$
(5,178)	1	[1,0,0]	$G(12, 4)$	$D_{12}$	(5,258)	8	[4,0,4]	$G(48, 51)$	$C_2^3 \times S_3$
(5,179)	1	[1,0,0]	$G(12, 4)$	$D_{12}$	(5,259)	4	[3,0,1]	$G(48, 51)$	$C_2^3 \times S_3$
(5,180)	1	[1,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,260)	4	[2,0,2]	$G(48, 51)$	$C_2^3 \times S_3$
(5,181)	2	[2,0,0]	$G(3, 1)$	$C_3$	(5,261)	8	[4,0,4]	$G(48, 51)$	$C_2^3 \times S_3$
(5,182)	2	[2,0,0]	$G(6, 2)$	$C_6$	(5,262)	5	[3,0,2]	$G(48, 51)$	$C_2^3 \times S_3$
(5,183)	1	[1,0,0]	$G(6, 2)$	$C_6$	(5,263)	5	[5,0,0]	$G(48, 51)$	$C_2^3 \times S_3$
(5,184)	1	[1,0,0]	$G(6, 2)$	$C_6$	(5,264)	5	[3,0,2]	$G(48, 51)$	$C_2^3 \times S_3$
(5,185)	3	[3,0,0]	$G(6, 1)$	$S_3$	(5,265)	10	[6,0,4]	$G(48, 51)$	$C_2^3 \times S_3$
(5,186)	3	[3,0,0]	$G(6, 1)$	$S_3$	(5,266)	4	[2,0,2]	$G(96, 230)$	$C_2^2 \times S_3$
(5,187)	4	[4,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,267)	4	[4,0,0]	$G(12, 2)$	$C_{12}$
(5,188)	8	[8,0,0]	$G(12, 4)$	$D_{12}$	(5,268)	4	[4,0,0]	$G(12, 2)$	$C_{12}$
(5,189)	4	[4,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,269)	2	[2,0,0]	$G(12, 2)$	$C_{12}$
(5,190)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,270)	2	[2,0,0]	$G(12, 2)$	$C_{12}$
(5,191)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,271)	6	[6,0,0]	$G(12, 1)$	$C_3 \times C_4$
(5,192)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,272)	6	[6,0,0]	$G(12, 1)$	$C_3 \times C_4$
(5,193)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,273)	2	[1,0,1]	$G(192, 1514)$	$C_2^2 \times S_3 \times D_4$
(5,194)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,274)	4	[2,0,2]	$G(24, 9)$	$C_{12} \times C_2$
(5,195)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,275)	8	[6,0,2]	$G(24, 10)$	$C_3 \times D_4$
(5,196)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,276)	4	[2,0,2]	$G(24, 10)$	$C_3 \times D_4$
(5,197)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,277)	4	[4,0,0]	$G(24, 10)$	$C_3 \times D_4$
(5,198)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,278)	2	[1,0,1]	$G(24, 10)$	$C_3 \times D_4$
(5,199)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,279)	4	[3,0,1]	$G(24, 10)$	$C_3 \times D_4$
(5,200)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,280)	4	[2,0,2]	$G(24, 10)$	$C_3 \times D_4$
(5,201)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,281)	4	[3,0,1]	$G(24, 10)$	$C_3 \times D_4$
(5,202)	2	[2,0,0]	$G(12, 4)$	$D_{12}$	(5,282)	4	[3,0,1]	$G(24, 10)$	$C_3 \times D_4$
(5,203)	2	[2,0,0]	$G(12, 4)$	$D_{12}$	(5,283)	4	[4,0,0]	$G(24, 10)$	$C_3 \times D_4$
(5,204)	4	[4,0,0]	$G(12, 4)$	$D_{12}$	(5,284)	2	[2,0,0]	$G(24, 10)$	$C_3 \times D_4$
(5,205)	6	[6,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,285)	2	[1,0,1]	$G(24, 9)$	$C_{12} \times C_2$
(5,206)	2	[2,0,0]	$G(24, 15)$	$C_6 \times C_2^2$	(5,286)	2	[2,0,0]	$G(24, 9)$	$C_{12} \times C_2$
(5,207)	6	[6,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,287)	2	[2,0,0]	$G(24, 9)$	$C_{12} \times C_2$
(5,208)	2	[2,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,288)	2	[1,0,1]	$G(24, 9)$	$C_{12} \times C_2$
(5,209)	4	[4,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,289)	2	[1,0,1]	$G(24, 9)$	$C_{12} \times C_2$
(5,210)	4	[4,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,290)	6	[3,0,3]	$G(24, 5)$	$C_4 \times S_3$
(5,211)	2	[2,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,291)	6	[3,0,3]	$G(24, 5)$	$C_4 \times S_3$
(5,212)	2	[2,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,292)	6	[6,0,0]	$G(24, 5)$	$C_4 \times S_3$
(5,213)	2	[2,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,293)	6	[6,0,0]	$G(24, 5)$	$C_4 \times S_3$
(5,214)	2	[2,0,0]	$G(24, 14)$	$C_2^2 \times S_3$	(5,294)	6	[3,0,3]	$G(24, 7)$	$C_2 \times (C_3 \times C_4)$
(5,215)	2	[2,0,0]	$G(48, 51)$	$C_2^3 \times S_3$	(5,295)	4	[2,0,2]	$G(24, 5)$	$C_4 \times S_3$
(5,216)	4	[4,0,0]	$G(6, 2)$	$C_6$	(5,296)	4	[2,0,2]	$G(24, 5)$	$C_4 \times S_3$
(5,217)	4	[4,0,0]	$G(6, 2)$	$C_6$	(5,297)	4	[4,0,0]	$G(24, 5)$	$C_4 \times S_3$
(5,218)	8	[8,0,0]	$G(6, 1)$	$S_3$	(5,298)	4	[4,0,0]	$G(24, 5)$	$C_4 \times S_3$
(5,219)	8	[8,0,0]	$G(6, 1)$	$S_3$	(5,299)	4	[2,0,2]	$G(24, 7)$	$C_2 \times (C_3 \times C_4)$
(5,220)	4	[4,0,0]	$G(6, 2)$	$C_6$	(5,300)	2	[2,0,0]	$G(24, 7)$	$C_2 \times (C_3 \times C_4)$
(5,221)	4	[4,0,0]	$G(6, 2)$	$C_6$	(5,301)	2	[2,0,0]	$G(24, 7)$	$C_2 \times (C_3 \times C_4)$
(5,222)	10	[10,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,302)	12	[9,0,3]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,223)	12	[12,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,303)	6	[3,0,3]	$G(24, 6)$	$D_{12}$
(5,224)	10	[8,0,2]	$G(12, 5)$	$C_6 \times C_2$	(5,304)	12	[9,0,3]	$G(24, 6)$	$D_{12}$
(5,225)	12	[12,0,0]	$G(12, 4)$	$D_{12}$	(5,305)	12	[6,0,6]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,226)	24	[24,0,0]	$G(12, 4)$	$D_{12}$	(5,306)	12	[9,0,3]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,227)	12	[10,0,2]	$G(12, 4)$	$D_{12}$	(5,307)	6	[6,0,0]	$G(24, 6)$	$D_{12}$
(5,228)	4	[3,0,1]	$G(12, 5)$	$C_6 \times C_2$	(5,308)	12	[12,0,0]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,229)	5	[5,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,309)	8	[4,0,4]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,230)	15	[15,0,0]	$G(12, 4)$	$D_{12}$	(5,310)	8	[6,0,2]	$G(24, 6)$	$D_{12}$
(5,231)	18	[18,0,0]	$G(12, 4)$	$D_{12}$	(5,311)	4	[2,0,2]	$G(24, 6)$	$D_{12}$
(5,232)	15	[12,0,3]	$G(12, 4)$	$D_{12}$	(5,312)	8	[6,0,2]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,233)	10	[6,0,4]	$G(24, 15)$	$C_6 \times C_2^2$	(5,313)	8	[8,0,0]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,234)	12	[8,0,4]	$G(24, 14)$	$C_2^2 \times S_3$	(5,314)	4	[4,0,0]	$G(24, 6)$	$D_{12}$
(5,235)	4	[2,0,2]	$G(24, 15)$	$C_6 \times C_2^2$	(5,315)	8	[6,0,2]	$G(24, 8)$	$(C_6 \times C_2) \times C_2$
(5,236)	4	[2,0,2]	$G(24, 15)$	$C_6 \times C_2^2$	(5,316)	4	[2,0,2]	$G(48, 45)$	$C_6 \times D_4$
(5,237)	4	[3,0,1]	$G(24, 15)$	$C_6 \times C_2^2$	(5,317)	4	[2,0,2]	$G(48, 45)$	$C_6 \times D_4$
(5,238)	5	[5,0,0]	$G(24, 15)$	$C_6 \times C_2^2$	(5,318)	2	[1,0,1]	$G(48, 45)$	$C_6 \times D_4$
(5,239)	5	[3,0,2]	$G(24, 15)$	$C_6 \times C_2^2$	(5,319)	4	[3,0,1]	$G(48, 45)$	$C_6 \times D_4$
(5,240)	15	[9,0,6]	$G(24, 14)$	$C_2^2 \times S_3$	(5,320)	2	[1,0,1]	$G(48, 45)$	$C_6 \times D_4$

Table 16 (continued): birational classification of the algebraic  $k$ -tori of dimension 5

CARAT	#	$[s, r, u]$	$G(n, i)$		CARAT	#	$[s, r, u]$	$G(n, i)$
(5,321)	2	[2,0,0]	$G(48, 45)$	$C_6 \times D_4$	(5,401)	2	[2,0,0]	$G(144, 192)$
(5,322)	2	[1,0,1]	$G(48, 45)$	$C_6 \times D_4$	(5,402)	2	[2,0,0]	$C_2^2 \times S_3^2$
(5,323)	4	[2,0,2]	$G(48, 45)$	$C_6 \times D_4$	(5,403)	1	[1,0,0]	$C_2^2 \times S_3^2$
(5,324)	2	[1,0,1]	$G(48, 45)$	$C_6 \times D_4$	(5,404)	5	[3,0,2]	$C_6 \times C_3$
(5,325)	2	[1,0,1]	$G(48, 44)$	$C_{12} \times C_2^2$	(5,405)	2	[2,0,0]	$C_6 \times C_3$
(5,326)	6	[3,0,3]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,406)	2	[2,0,0]	$C_6 \times C_3$
(5,327)	2	[1,0,1]	$G(48, 42)$	$C_2^2 \times (C_3 \times C_4)$	(5,407)	4	[3,0,1]	$C_3 \times S_3$
(5,328)	4	[2,0,2]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,408)	4	[3,0,1]	$C_3 \times S_3$
(5,329)	2	[2,0,0]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,409)	9	[6,0,3]	$C_2^3 \times C_2$
(5,330)	2	[2,0,0]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,410)	9	[6,0,3]	$C_3^3 \times C_2$
(5,331)	2	[1,0,1]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,411)	2	[1,0,1]	$C_6 \times C_3$
(5,332)	2	[1,0,1]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,412)	2	[1,0,1]	$C_6 \times C_3$
(5,333)	4	[2,0,2]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,413)	8	[6,0,2]	$C_3 \times S_3$
(5,334)	4	[2,0,2]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,414)	8	[6,0,2]	$C_3 \times S_3$
(5,335)	6	[3,0,3]	$G(48, 36)$	$C_2 \times D_{12}$	(5,415)	2	[2,0,0]	$C_6 \times C_2^2$
(5,336)	12	[6,0,6]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,416)	6	[6,0,0]	$C_2^2 \times S_3$
(5,337)	12	[9,0,3]	$G(48, 38)$	$D_4 \times S_3$	(5,417)	2	[2,0,0]	$C_2^3 \times S_3$
(5,338)	6	[3,0,3]	$G(48, 38)$	$D_4 \times S_3$	(5,418)	2	[2,0,0]	$C_2^2 \times S_3$
(5,339)	6	[3,0,3]	$G(48, 38)$	$D_4 \times S_3$	(5,419)	4	[4,0,0]	$C_2^3 \times S_3$
(5,340)	12	[6,0,6]	$G(48, 38)$	$D_4 \times S_3$	(5,420)	4	[4,0,0]	$C_2^3 \times S_3$
(5,341)	6	[6,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,421)	1	[1,0,0]	$C_2^3 \times S_3^2$
(5,342)	6	[3,0,3]	$G(48, 38)$	$D_4 \times S_3$	(5,422)	2	[2,0,0]	$C_6 \times C_6$
(5,343)	8	[4,0,4]	$G(48, 38)$	$D_4 \times S_3$	(5,423)	4	[3,0,1]	$C_6 \times S_3$
(5,344)	8	[6,0,2]	$G(48, 38)$	$D_4 \times S_3$	(5,424)	9	[6,0,3]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,345)	4	[2,0,2]	$G(48, 38)$	$D_4 \times S_3$	(5,425)	2	[1,0,1]	$C_6 \times C_6$
(5,346)	8	[4,0,4]	$G(48, 38)$	$D_4 \times S_3$	(5,426)	8	[6,0,2]	$C_6 \times S_3$
(5,347)	8	[4,0,4]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,427)	4	[4,0,0]	$C_6 \times S_3$
(5,348)	8	[4,0,4]	$G(48, 38)$	$D_4 \times S_3$	(5,428)	2	[2,0,0]	$C_6 \times S_3$
(5,349)	4	[2,0,2]	$G(48, 38)$	$D_4 \times S_3$	(5,429)	2	[2,0,0]	$C_6 \times S_3$
(5,350)	8	[4,0,4]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,430)	3	[2,0,1]	$C_6 \times S_3$
(5,351)	4	[2,0,2]	$G(48, 36)$	$C_2 \times D_{12}$	(5,431)	3	[2,0,1]	$C_6 \times S_3$
(5,352)	4	[3,0,1]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,432)	3	[2,0,1]	$C_6 \times S_3$
(5,353)	4	[2,0,2]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,433)	3	[2,0,1]	$C_6 \times S_3$
(5,354)	4	[3,0,1]	$G(48, 36)$	$C_2 \times D_{12}$	(5,434)	8	[6,0,2]	$S_3^2$
(5,355)	2	[1,0,1]	$G(48, 36)$	$C_2 \times D_{12}$	(5,435)	11	[9,0,2]	$S_3^2$
(5,356)	2	[2,0,0]	$G(48, 36)$	$C_2 \times D_{12}$	(5,436)	8	[6,0,2]	$S_3^2$
(5,357)	4	[3,0,1]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,437)	1	[1,0,0]	$C_6 \times C_6$
(5,358)	4	[4,0,0]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,438)	1	[1,0,0]	$C_6 \times C_6$
(5,359)	8	[6,0,2]	$G(48, 38)$	$D_4 \times S_3$	(5,439)	1	[1,0,0]	$C_6 \times C_6$
(5,360)	8	[4,0,4]	$G(48, 38)$	$D_4 \times S_3$	(5,440)	3	[3,0,0]	$C_6 \times S_3$
(5,361)	8	[4,0,4]	$G(48, 38)$	$D_4 \times S_3$	(5,441)	3	[3,0,0]	$C_6 \times S_3$
(5,362)	4	[2,0,2]	$G(48, 36)$	$C_2 \times D_{12}$	(5,442)	3	[3,0,0]	$C_6 \times S_3$
(5,363)	4	[2,0,2]	$G(48, 36)$	$C_2 \times D_{12}$	(5,443)	3	[3,0,0]	$C_6 \times S_3$
(5,364)	8	[4,0,4]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,444)	7	[6,0,1]	$S_3^2$
(5,365)	8	[6,0,2]	$G(48, 38)$	$D_4 \times S_3$	(5,445)	7	[6,0,1]	$S_3^2$
(5,366)	8	[8,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,446)	7	[6,0,1]	$S_3^2$
(5,367)	4	[4,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,447)	7	[6,0,1]	$S_3^2$
(5,368)	4	[2,0,2]	$G(48, 38)$	$D_4 \times S_3$	(5,448)	3	[2,0,1]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,369)	8	[4,0,4]	$G(48, 38)$	$D_4 \times S_3$	(5,449)	3	[2,0,1]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,370)	2	[1,0,1]	$G(96, 221)$	$C_2 \times C_6 \times D_4$	(5,450)	5	[3,0,2]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,371)	2	[1,0,1]	$G(96, 206)$	$C_2^2 \times C_4 \times S_3$	(5,451)	2	[2,0,0]	$C_6 \times S_3$
(5,372)	6	[3,0,3]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,452)	1	[1,0,0]	$C_6 \times S_3$
(5,373)	8	[4,0,4]	$G(96, 209)$	$C_2 \times D_4 \times S_3$	(5,453)	1	[1,0,0]	$C_6 \times S_3$
(5,374)	4	[2,0,2]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,454)	3	[3,0,0]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,375)	2	[2,0,0]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,455)	3	[3,0,0]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,376)	4	[3,0,1]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,456)	6	[6,0,0]	$C_2 \times (C_2^3 \rtimes C_2)$
(5,377)	4	[2,0,2]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,457)	5	[4,0,1]	$S_3^2$
(5,378)	4	[2,0,2]	$G(96, 219)$	$C_2^2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,458)	4	[3,0,1]	$S_3^2$
(5,379)	2	[1,0,1]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,459)	4	[3,0,1]	$S_3^2$
(5,380)	8	[4,0,4]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,460)	2	[2,0,0]	$C_2^3 \times S_3$
(5,381)	4	[2,0,2]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,461)	4	[4,0,0]	$C_6$
(5,382)	2	[1,0,1]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,462)	4	[4,0,0]	$C_6$
(5,383)	2	[1,0,1]	$G(96, 207)$	$C_2^2 \times D_{12}$	(5,463)	2	[2,0,0]	$C_2 \times C_6 \times S_3$
(5,384)	4	[2,0,2]	$G(96, 209)$	$C_2 \times D_4 \times S_3$	(5,464)	3	[2,0,1]	$C_2 \times C_6 \times S_3$
(5,385)	2	[1,0,1]	$G(96, 209)$	$C_2 \times S_3 \times D_4$	(5,465)	8	[6,0,2]	$C_2 \times S_3^2$
(5,386)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,466)	1	[1,0,0]	$C_6 \times C_6 \times C_2$
(5,387)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,467)	3	[3,0,0]	$C_2 \times C_6 \times S_3$
(5,388)	2	[2,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,468)	7	[6,0,1]	$C_2 \times S_3^2$
(5,389)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,469)	3	[2,0,1]	$C_2^2 \times (C_2^3 \rtimes C_2)$
(5,390)	6	[6,0,0]	$G(12, 4)$	$D_{12}$	(5,470)	1	[1,0,0]	$C_2 \times C_6 \times S_3$
(5,391)	12	[12,0,0]	$G(12, 4)$	$D_{12}$	(5,471)	1	[1,0,0]	$C_2 \times C_6 \times S_3$
(5,392)	4	[4,0,0]	$G(12, 5)$	$C_6 \times C_2$	(5,472)	2	[2,0,0]	$C_2 \times C_6 \times S_3$
(5,393)	1	[1,0,0]	$G(144, 195)$	$C_2^2 \times C_6 \times S_3$	(5,473)	1	[1,0,0]	$C_2 \times C_6 \times S_3$
(5,394)	3	[3,0,0]	$G(144, 192)$	$C_2^2 \times S_3^2$	(5,474)	1	[1,0,0]	$C_2 \times C_6 \times S_3$
(5,395)	4	[3,0,1]	$G(144, 192)$	$C_2^3 \times S_3^2$	(5,475)	1	[1,0,0]	$C_2 \times C_6 \times S_3$
(5,396)	1	[1,0,0]	$G(144, 196)$	$C_2^3 \times (C_2^3 \rtimes C_2)$	(5,476)	2	[2,0,0]	$C_2 \times C_6 \times S_3$
(5,397)	1	[1,0,0]	$G(144, 192)$	$C_2^3 \times S_3^2$	(5,477)	3	[3,0,0]	$C_2^2 \times (C_2^3 \rtimes C_2)$
(5,398)	1	[1,0,0]	$G(144, 192)$	$C_2^3 \times S_3^2$	(5,478)	3	[3,0,0]	$C_2 \times S_3^2$
(5,399)	3	[3,0,0]	$G(144, 192)$	$C_2^2 \times S_3^2$	(5,479)	6	[6,0,0]	$C_2 \times S_3^2$
(5,400)	2	[2,0,0]	$G(144, 192)$	$C_2^2 \times S_3^2$	(5,480)	6	[6,0,0]	$C_2 \times S_3^2$



Table 16 (continued): birational classification of the algebraic  $k$ -tori of dimension 5

CARAT	#	$[s, r, u]$	$G(n, i)$		CARAT	#	$[s, r, u]$	$G(n, i)$	
(5,481)	3	[3,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,561)	5	[1,0,4]	$G(48, 31)$	$C_4 \times A_4$
(5,482)	3	[3,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,562)	5	[1,0,4]	$G(48, 30)$	$A_4 \rtimes C_4$
(5,483)	3	[3,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,563)	5	[2,0,3]	$G(48, 30)$	$A_4 \rtimes C_4$
(5,484)	5	[4,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,564)	5	[1,0,4]	$G(96, 197)$	$D_4 \times A_4$
(5,485)	4	[3,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,565)	10	[2,0,8]	$G(96, 197)$	$D_4 \times A_4$
(5,486)	4	[3,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,566)	5	[2,0,3]	$G(96, 197)$	$A_4 \times D_4$
(5,487)	4	[3,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,567)	5	[1,0,4]	$G(96, 196)$	$C_2 \times C_4 \times A_4$
(5,488)	5	[4,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,568)	10	[2,0,8]	$G(96, 195)$	$(C_2^2 \times A_4) \rtimes C_2$
(5,489)	4	[3,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,569)	10	[2,0,8]	$G(96, 195)$	$(C_2^2 \times A_4) \rtimes C_2$
(5,490)	4	[3,0,1]	$G(72, 46)$	$C_2 \times S_3^2$	(5,570)	10	[2,0,8]	$G(96, 195)$	$(C_2^2 \times A_4) \rtimes C_2$
(5,491)	2	[2,0,0]	$G(72, 49)$	$C_2^2 \times (C_3^2 \times C_2)$	(5,571)	10	[4,0,6]	$G(96, 195)$	$(C_2^2 \times A_4) \rtimes C_2$
(5,492)	1	[1,0,0]	$G(72, 49)$	$C_2^2 \times (C_3^2 \times C_2)$	(5,572)	5	[2,0,3]	$G(96, 187)$	$(C_2 \times S_4) \rtimes C_2$
(5,493)	1	[1,0,0]	$G(72, 49)$	$C_2^2 \times (C_3^2 \times C_2)$	(5,573)	5	[1,0,4]	$G(96, 187)$	$(C_2 \times S_4) \rtimes C_2$
(5,494)	2	[2,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,574)	10	[2,0,8]	$G(96, 187)$	$(C_2 \times S_4) \rtimes C_2$
(5,495)	2	[2,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,575)	5	[1,0,4]	$G(96, 186)$	$C_4 \times S_4$
(5,496)	2	[2,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,576)	5	[2,0,3]	$G(96, 186)$	$C_4 \times S_4$
(5,497)	2	[2,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,577)	5	[1,0,4]	$G(96, 186)$	$C_4 \times S_4$
(5,498)	4	[4,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,578)	5	[1,0,4]	$G(96, 186)$	$C_4 \times S_4$
(5,499)	4	[4,0,0]	$G(72, 46)$	$C_2 \times S_3^2$	(5,579)	5	[1,0,4]	$G(96, 194)$	$C_2 \times (A_4 \times C_4)$
(5,500)	2	[2,0,0]	$G(72, 49)$	$C_2^2 \times (C_3^2 \times C_2)$	(5,580)	5	[3,0,2]	$G(12, 3)$	$A_4$
(5,501)	5	[3,0,2]	$G(9, 2)$	$C_3 \times C_3$	(5,581)	3	[1,0,2]	$G(144, 193)$	$C_2 \times C_6 \times A_4$
(5,502)	6	[5,0,1]	$G(12, 3)$	$A_4$	(5,582)	3	[1,0,2]	$G(144, 188)$	$C_6 \times S_4$
(5,503)	6	[2,0,4]	$G(24, 13)$	$C_2 \times A_4$	(5,583)	3	[1,0,2]	$G(144, 188)$	$C_6 \times S_4$
(5,504)	5	[2,0,3]	$G(24, 13)$	$C_2 \times A_4$	(5,584)	3	[1,0,2]	$G(144, 188)$	$C_6 \times S_4$
(5,505)	5	[2,0,3]	$G(24, 13)$	$C_2 \times A_4$	(5,585)	3	[1,0,2]	$G(144, 188)$	$C_6 \times S_4$
(5,506)	6	[5,0,1]	$G(24, 12)$	$S_4$	(5,586)	3	[2,0,1]	$G(144, 188)$	$C_6 \times S_4$
(5,507)	6	[3,0,3]	$G(24, 12)$	$S_4$	(5,587)	3	[1,0,2]	$G(144, 188)$	$C_6 \times S_4$
(5,508)	5	[2,0,3]	$G(24, 12)$	$S_4$	(5,588)	3	[1,0,2]	$G(144, 190)$	$C_2 \times A_4 \times S_3$
(5,509)	5	[3,0,2]	$G(24, 12)$	$S_4$	(5,589)	6	[2,0,4]	$G(144, 190)$	$C_2 \times A_4 \times S_3$
(5,510)	5	[1,0,4]	$G(48, 49)$	$C_2^2 \times A_4$	(5,590)	3	[2,0,1]	$G(144, 190)$	$C_2 \times A_4 \times S_3$
(5,511)	6	[2,0,4]	$G(48, 48)$	$C_2 \times S_4$	(5,591)	6	[2,0,4]	$G(144, 190)$	$C_2 \times A_4 \times S_3$
(5,512)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$	(5,592)	6	[2,0,4]	$G(144, 183)$	$S_4 \times S_3$
(5,513)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$	(5,593)	6	[2,0,4]	$G(144, 183)$	$S_4 \times S_3$
(5,514)	5	[1,0,4]	$G(48, 48)$	$C_2 \times S_4$	(5,594)	6	[2,0,4]	$G(144, 189)$	$C_2 \times ((C_3 \times A_4) \rtimes C_2)$
(5,515)	5	[1,0,4]	$G(48, 48)$	$C_2 \times S_4$	(5,595)	6	[2,0,4]	$G(144, 183)$	$S_4 \times S_3$
(5,516)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$	(5,596)	6	[4,0,2]	$G(144, 183)$	$S_4 \times S_3$
(5,517)	5	[1,0,4]	$G(96, 226)$	$C_2^2 \times S_4$	(5,597)	6	[2,0,4]	$G(144, 183)$	$S_4 \times S_3$
(5,518)	12	[2,0,10]	$G(192, 1537)$	$C_3^3 \times S_4$	(5,598)	6	[4,0,2]	$G(144, 183)$	$S_4 \times S_3$
(5,519)	17	[6,0,11]	$G(24, 13)$	$C_2 \times A_4$	(5,599)	6	[2,0,4]	$G(144, 183)$	$S_4 \times S_3$
(5,520)	17	[10,0,7]	$G(24, 13)$	$C_2 \times A_4$	(5,600)	6	[2,0,4]	$G(144, 189)$	$C_2 \times ((C_3 \times A_4) \rtimes C_2)$
(5,521)	17	[12,0,5]	$G(24, 12)$	$S_4$	(5,601)	3	[2,0,1]	$G(144, 189)$	$C_2 \times ((C_3 \times A_4) \rtimes C_2)$
(5,522)	17	[8,0,9]	$G(24, 12)$	$S_4$	(5,602)	3	[1,0,2]	$G(144, 189)$	$C_2 \times ((C_3 \times A_4) \rtimes C_2)$
(5,523)	17	[3,0,14]	$G(48, 49)$	$C_2^2 \times A_4$	(5,603)	6	[2,0,4]	$G(144, 183)$	$S_4 \times S_3$
(5,524)	14	[4,0,10]	$G(48, 49)$	$C_2^2 \times A_4$	(5,604)	5	[1,0,4]	$G(24, 13)$	$C_2 \times A_4$
(5,525)	12	[4,0,8]	$G(48, 49)$	$C_2^2 \times A_4$	(5,605)	5	[2,0,3]	$G(24, 13)$	$C_2 \times A_4$
(5,526)	12	[2,0,10]	$G(48, 49)$	$C_2^2 \times A_4$	(5,606)	5	[2,0,3]	$G(24, 13)$	$C_2 \times A_4$
(5,527)	17	[3,0,14]	$G(48, 48)$	$C_2 \times S_4$	(5,607)	10	[6,0,4]	$G(24, 12)$	$S_4$
(5,528)	17	[4,0,13]	$G(48, 48)$	$C_2 \times S_4$	(5,608)	10	[4,0,6]	$G(24, 12)$	$S_4$
(5,529)	17	[6,0,11]	$G(48, 48)$	$C_2 \times S_4$	(5,609)	3	[1,0,2]	$G(288, 1033)$	$C_2 \times C_6 \times S_4$
(5,530)	17	[4,0,13]	$G(48, 48)$	$C_2 \times S_4$	(5,610)	3	[1,0,2]	$G(288, 1037)$	$C_2^2 \times A_4 \times S_3$
(5,531)	17	[10,0,7]	$G(48, 48)$	$C_2 \times S_4$	(5,611)	3	[1,0,2]	$G(288, 1034)$	$C_2^2 \times ((C_3 \times A_4) \rtimes C_2)$
(5,532)	12	[4,0,8]	$G(48, 48)$	$C_2 \times S_4$	(5,612)	6	[2,0,4]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,533)	14	[6,0,8]	$G(48, 48)$	$C_2 \times S_4$	(5,613)	6	[2,0,4]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,534)	12	[2,0,10]	$G(48, 48)$	$C_2 \times S_4$	(5,614)	3	[1,0,2]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,535)	14	[5,0,9]	$G(48, 48)$	$C_2 \times S_4$	(5,615)	3	[2,0,1]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,536)	14	[3,0,11]	$G(48, 48)$	$C_2 \times S_4$	(5,616)	3	[1,0,2]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,537)	14	[4,0,10]	$G(48, 48)$	$C_2 \times S_4$	(5,617)	6	[2,0,4]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,538)	14	[4,0,10]	$G(48, 48)$	$C_2 \times S_4$	(5,618)	6	[2,0,4]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,539)	12	[2,0,10]	$G(96, 228)$	$C_3^2 \times A_4$	(5,619)	3	[1,0,2]	$G(288, 1028)$	$C_2 \times S_4 \times S_3$
(5,540)	17	[3,0,14]	$G(96, 226)$	$C_3^2 \times S_4$	(5,620)	3	[2,0,1]	$G(36, 11)$	$C_3 \times A_4$
(5,541)	14	[4,0,10]	$G(96, 226)$	$C_3^2 \times S_4$	(5,621)	5	[1,0,4]	$G(48, 49)$	$C_2^2 \times A_4$
(5,542)	14	[2,0,12]	$G(96, 226)$	$C_3^2 \times S_4$	(5,622)	5	[1,0,4]	$G(48, 48)$	$C_2 \times S_4$
(5,543)	12	[4,0,8]	$G(96, 226)$	$C_3^2 \times S_4$	(5,623)	10	[4,0,6]	$G(48, 48)$	$C_2 \times S_4$
(5,544)	12	[2,0,10]	$G(96, 226)$	$C_3^2 \times S_4$	(5,624)	10	[2,0,8]	$G(48, 48)$	$C_2 \times S_4$
(5,545)	12	[2,0,10]	$G(96, 226)$	$C_3^2 \times S_4$	(5,625)	5	[2,0,3]	$G(48, 48)$	$C_2 \times S_4$
(5,546)	12	[2,0,10]	$G(96, 226)$	$C_3^2 \times S_4$	(5,626)	3	[1,0,2]	$G(576, 8659)$	$C_2^2 \times S_4 \times S_3$
(5,547)	12	[2,0,10]	$G(96, 226)$	$C_3^2 \times S_4$	(5,627)	3	[1,0,2]	$G(72, 47)$	$C_6 \times A_4$
(5,548)	14	[3,0,11]	$G(96, 226)$	$C_3^2 \times S_4$	(5,628)	3	[1,0,2]	$G(72, 47)$	$C_6 \times A_4$
(5,549)	5	[1,0,4]	$G(192, 1497)$	$C_2 \times D_4 \times A_4$	(5,629)	3	[2,0,1]	$G(72, 47)$	$C_6 \times A_4$
(5,550)	10	[2,0,8]	$G(192, 1488)$	$C_2 \times ((C_2^2 \times A_4) \rtimes C_2)$	(5,630)	3	[1,0,2]	$G(72, 42)$	$C_3 \times S_4$
(5,551)	10	[2,0,8]	$G(192, 1472)$	$D_4 \times S_4$	(5,631)	3	[1,0,2]	$G(72, 42)$	$C_3 \times S_4$
(5,552)	10	[2,0,8]	$G(192, 1472)$	$D_4 \times S_4$	(5,632)	3	[2,0,1]	$G(72, 42)$	$C_3 \times S_4$
(5,553)	5	[1,0,4]	$G(192, 1472)$	$D_4 \times S_4$	(5,633)	3	[2,0,1]	$G(72, 42)$	$C_3 \times S_4$
(5,554)	5	[1,0,4]	$G(192, 1472)$	$D_4 \times S_4$	(5,634)	6	[4,0,2]	$G(72, 44)$	$A_4 \times S_3$
(5,555)	5	[2,0,3]	$G(192, 1472)$	$S_4 \times D_4$	(5,635)	6	[2,0,4]	$G(72, 44)$	$A_4 \times S_3$
(5,556)	5	[1,0,4]	$G(192, 1472)$	$S_4 \times D_4$	(5,636)	6	[2,0,4]	$G(72, 43)$	$(C_3 \times A_4) \rtimes C_2$
(5,557)	5	[1,0,4]	$G(192, 1470)$	$C_2 \times ((C_2 \times S_4) \rtimes C_2)$	(5,637)	6	[4,0,2]	$G(72, 43)$	$(C_3 \times A_4) \rtimes C_2$
(5,558)	5	[1,0,4]	$G(192, 1469)$	$C_2 \times C_4 \times S_4$	(5,638)	5	[1,0,4]	$G(96, 226)$	$C_2^2 \times S_4$
(5,559)	5	[1,0,4]	$G(384, 20051)$	$C_2 \times D_4 \times S_4$	(5,639)	2	[1,0,1]	$G(16, 5)$	$C_8 \times C_2$
(5,560)	5	[2,0,3]	$G(48, 31)$	$C_4 \times A_4$	(5,640)	2	[2,0,0]	$G(16, 7)$	$D_8$

Table 16 (continued): birational classification of the algebraic  $k$ -tori of dimension 5

CARAT	#	$[s, r, u]$	$G(n, i)$		CARAT	#	$[s, r, u]$	$G(n, i)$	
(5,641)	4	[3,0,1]	$G(16, 7)$	$D_8$	(5,721)	4	[1,0,3]	$G(32, 40)$	$C_2 \times QD_8$
(5,642)	2	[1,0,1]	$G(16, 7)$	$D_8$	(5,722)	1	[0,0,1]	$G(384, 18101)$	$C_2 \times ((SL(2, 3) \rtimes C_4) \rtimes C_2)$
(5,643)	2	[1,0,1]	$G(32, 39)$	$C_2 \times D_8$	(5,723)	5	[1,0,4]	$G(384, 5833)$	
(5,644)	2	[2,0,0]	$G(8, 1)$	$C_8$	(5,724)	5	[1,0,4]	$G(384, 5602)$	
(5,645)	2	[2,0,0]	$G(8, 1)$	$C_8$	(5,725)	5	[1,0,4]	$G(384, 5602)$	
(5,646)	2	[0,0,2]	$G(1152, 157511)$		(5,726)	5	[2,0,3]	$G(384, 5602)$	
(5,647)	2	[0,0,2]	$G(1152, 157478)$		(5,727)	5	[1,0,4]	$G(384, 20090)$	
(5,648)	1	[0,0,1]	$G(1152, 157478)$		(5,728)	5	[1,0,4]	$G(384, 5602)$	
(5,649)	1	[0,0,1]	$G(1152, 157528)$		(5,729)	5	[1,0,4]	$G(384, 20089)$	
(5,650)	1	[0,0,1]	$G(1152, 157478)$		(5,730)	1	[0,0,1]	$G(48, 32)$	$C_2 \times SL(2, 3)$
(5,651)	8	[2,0,6]	$G(128, 928)$	$D_4^2 \rtimes C_2$	(5,731)	1	[0,0,1]	$G(48, 33)$	$SL(2, 3) \rtimes C_2$
(5,652)	4	[1,0,3]	$G(128, 928)$	$D_4^2 \rtimes C_2$	(5,732)	1	[0,0,1]	$G(48, 33)$	$SL(2, 3) \rtimes C_2$
(5,653)	8	[2,0,6]	$G(128, 928)$	$D_4^2 \rtimes C_2$	(5,733)	1	[0,0,1]	$G(48, 29)$	$GL(2, 3)$
(5,654)	4	[1,0,3]	$G(128, 928)$	$D_4^2 \rtimes C_2$	(5,734)	1	[0,0,1]	$G(48, 29)$	$GL(2, 3)$
(5,655)	4	[2,0,2]	$G(128, 928)$	$D_4^2 \rtimes C_2$	(5,735)	1	[0,0,1]	$G(48, 46)$	$C_6 \times Q_8$
(5,656)	4	[2,0,2]	$G(128, 928)$	$D_4^2 \rtimes C_2$	(5,736)	1	[0,1,0]	$G(48, 9)$	$C_2 \times (C_3 \rtimes C_8)$
(5,657)	8	[2,0,6]	$G(128, 1755)$		(5,737)	1	[0,0,1]	$G(48, 17)$	$(C_3 \times Q_8) \rtimes C_2$
(5,658)	4	[1,0,3]	$G(128, 1746)$	$C_2 \times ((C_4^2 \rtimes C_2) \rtimes C_2)$	(5,738)	1	[0,0,1]	$G(48, 17)$	$(C_3 \times Q_8) \rtimes C_2$
(5,659)	8	[2,0,6]	$G(128, 850)$		(5,739)	1	[0,0,1]	$G(48, 17)$	$(C_3 \times Q_8) \rtimes C_2$
(5,660)	4	[1,0,3]	$G(128, 856)$		(5,740)	1	[0,0,1]	$G(48, 17)$	$(C_3 \times Q_8) \rtimes C_2$
(5,661)	1	[0,0,1]	$G(144, 156)$	$C_6 \times SL(2, 3)$	(5,741)	5	[1,0,4]	$G(48, 32)$	$C_2 \times SL(2, 3)$
(5,662)	1	[0,0,1]	$G(144, 125)$	$(C_3 \times SL(2, 3)) \rtimes C_2$	(5,742)	5	[2,0,3]	$G(48, 29)$	$GL(2, 3)$
(5,663)	1	[0,0,1]	$G(144, 125)$	$(C_3 \times SL(2, 3)) \rtimes C_2$	(5,743)	5	[1,0,4]	$G(48, 29)$	$GL(2, 3)$
(5,664)	4	[1,0,3]	$G(16, 12)$	$C_2 \times Q_8$	(5,744)	1	[0,0,1]	$G(576, 8359)$	
(5,665)	4	[2,0,2]	$G(16, 6)$	$C_8 \rtimes C_2$	(5,745)	2	[0,0,2]	$G(576, 8277)$	
(5,666)	4	[2,0,2]	$G(16, 6)$	$C_8 \rtimes C_2$	(5,746)	2	[0,0,2]	$G(576, 8277)$	
(5,667)	4	[1,0,3]	$G(16, 6)$	$C_8 \rtimes C_2$	(5,747)	1	[0,0,1]	$G(576, 8282)$	
(5,668)	4	[2,0,2]	$G(16, 13)$	$(C_4 \times C_2) \rtimes C_2$	(5,748)	1	[0,0,1]	$G(576, 8282)$	
(5,669)	4	[1,0,3]	$G(16, 13)$	$(C_4 \times C_2) \rtimes C_2$	(5,749)	4	[1,0,3]	$G(64, 101)$	$C_2 \times (C_4^2 \rtimes C_2)$
(5,670)	7	[3,0,4]	$G(16, 13)$	$(C_4 \times C_2) \rtimes C_2$	(5,750)	8	[4,0,4]	$G(64, 138)$	
(5,671)	7	[3,0,4]	$G(16, 13)$	$(C_4 \times C_2) \rtimes C_2$	(5,751)	8	[2,0,6]	$G(64, 138)$	
(5,672)	4	[2,0,2]	$G(16, 8)$	$QD_8$	(5,752)	8	[2,0,6]	$G(64, 34)$	
(5,673)	4	[1,0,3]	$G(16, 8)$	$QD_8$	(5,753)	4	[2,0,2]	$G(64, 34)$	
(5,674)	4	[2,0,2]	$G(16, 8)$	$QD_8$	(5,754)	8	[4,0,4]	$G(64, 32)$	$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$
(5,675)	4	[1,0,3]	$G(16, 8)$	$QD_8$	(5,755)	8	[2,0,6]	$G(64, 32)$	$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$
(5,676)	1	[0,0,1]	$G(192, 988)$	$(SL(2, 3) \rtimes C_4) \rtimes C_2$	(5,756)	4	[2,0,2]	$G(64, 134)$	$(C_4^2 \rtimes C_2) \rtimes C_2$
(5,677)	1	[0,0,1]	$G(192, 988)$	$(SL(2, 3) \rtimes C_4) \rtimes C_2$	(5,757)	4	[1,0,3]	$G(64, 134)$	$(C_4^2 \rtimes C_2) \rtimes C_2$
(5,678)	1	[0,0,1]	$G(192, 988)$	$(SL(2, 3) \rtimes C_4) \rtimes C_2$	(5,758)	4	[2,0,2]	$G(64, 134)$	$(C_4^2 \rtimes C_2) \rtimes C_2$
(5,679)	1	[0,0,1]	$G(192, 988)$	$(SL(2, 3) \rtimes C_4) \rtimes C_2$	(5,759)	4	[1,0,3]	$G(64, 134)$	$(C_4^2 \rtimes C_2) \rtimes C_2$
(5,680)	1	[0,0,1]	$G(192, 981)$	$C_2 \times (SL(2, 3) \rtimes C_4)$	(5,760)	8	[4,0,4]	$G(64, 32)$	$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$
(5,681)	1	[0,0,1]	$G(192, 1502)$	$C_2 \times ((SL(2, 3) \rtimes C_2) \rtimes C_2)$	(5,761)	8	[2,0,6]	$G(64, 32)$	$((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$
(5,682)	1	[0,0,1]	$G(192, 1481)$	$C_2 \times (GL(2, 3) \rtimes C_2)$	(5,762)	4	[2,0,2]	$G(64, 34)$	
(5,683)	5	[1,0,4]	$G(192, 201)$		(5,763)	11	[5,0,6]	$G(64, 138)$	
(5,684)	5	[2,0,3]	$G(192, 201)$		(5,764)	11	[3,0,8]	$G(64, 138)$	
(5,685)	5	[1,0,4]	$G(192, 1508)$		(5,765)	4	[1,0,3]	$G(64, 134)$	$(C_4^2 \rtimes C_2) \rtimes C_2$
(5,686)	5	[1,0,4]	$G(192, 1494)$		(5,766)	4	[1,0,3]	$G(64, 134)$	$(C_4^2 \rtimes C_2) \rtimes C_2$
(5,687)	5	[2,0,3]	$G(192, 1494)$		(5,767)	4	[1,0,3]	$G(64, 264)$	$C_2 \times ((C_2 \times D_4) \rtimes C_2)$
(5,688)	5	[1,0,4]	$G(192, 1493)$		(5,768)	11	[3,0,8]	$G(64, 90)$	
(5,689)	5	[2,0,3]	$G(192, 1493)$		(5,769)	4	[1,0,3]	$G(64, 92)$	$C_2 \times ((C_8 \rtimes C_2) \rtimes C_2)$
(5,690)	1	[0,0,1]	$G(2304, ?)$		(5,770)	4	[1,0,3]	$G(64, 254)$	$C_2 \times ((C_2 \times D_4) \rtimes C_2)$
(5,691)	1	[0,0,1]	$G(24, 3)$	$SL(2, 3)$	(5,771)	1	[0,0,1]	$G(72, 25)$	$C_3 \times SL(2, 3)$
(5,692)	1	[0,1,0]	$G(24, 1)$	$C_3 \times C_8$	(5,772)	5	[1,0,4]	$G(768, 1088539)$	
(5,693)	1	[0,1,0]	$G(24, 1)$	$C_3 \times C_8$	(5,773)	4	[2,0,2]	$G(8, 4)$	$Q_8$
(5,694)	1	[0,0,1]	$G(24, 11)$	$C_3 \times Q_8$	(5,774)	4	[2,0,2]	$G(8, 4)$	$Q_8$
(5,695)	1	[0,0,1]	$G(24, 11)$	$C_3 \times Q_8$	(5,775)	1	[0,0,1]	$G(96, 200)$	$C_2 \times (SL(2, 3) \rtimes C_2)$
(5,696)	5	[2,0,3]	$G(24, 3)$	$SL(2, 3)$	(5,776)	1	[0,0,1]	$G(96, 189)$	$C_2 \times GL(2, 3)$
(5,697)	4	[1,0,3]	$G(256, 26531)$	$C_2 \times (D_4^2 \rtimes C_2)$	(5,777)	1	[0,0,1]	$G(96, 67)$	$SL(2, 3) \rtimes C_4$
(5,698)	1	[0,0,1]	$G(288, 860)$	$((SL(2, 3) \rtimes C_2) \rtimes C_2) \rtimes C_3$	(5,778)	1	[0,0,1]	$G(96, 67)$	$SL(2, 3) \rtimes C_4$
(5,699)	1	[0,0,1]	$G(288, 911)$	$C_2 \times ((C_3 \times SL(2, 3)) \rtimes C_2)$	(5,779)	1	[0,0,1]	$G(96, 201)$	$(SL(2, 3) \rtimes C_2) \rtimes C_2$
(5,700)	4	[2,0,2]	$G(32, 11)$	$C_4^2 \rtimes C_2$	(5,780)	1	[0,0,1]	$G(96, 193)$	$GL(2, 3) \rtimes C_2$
(5,701)	4	[1,0,3]	$G(32, 11)$	$C_4^2 \rtimes C_2$	(5,781)	1	[0,0,1]	$G(96, 193)$	$GL(2, 3) \rtimes C_2$
(5,702)	4	[1,0,3]	$G(32, 11)$	$C_4^2 \rtimes C_2$	(5,782)	1	[0,0,1]	$G(96, 201)$	$(SL(2, 3) \rtimes C_2) \rtimes C_2$
(5,703)	4	[2,0,2]	$G(32, 11)$	$C_4^2 \rtimes C_2$	(5,783)	1	[0,0,1]	$G(96, 193)$	$GL(2, 3) \rtimes C_2$
(5,704)	11	[5,0,6]	$G(32, 6)$	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	(5,784)	1	[0,0,1]	$G(96, 148)$	$C_2 \times ((C_3 \times Q_8) \rtimes C_2)$
(5,705)	11	[6,0,5]	$G(32, 6)$	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	(5,785)	5	[2,0,3]	$G(96, 204)$	$((C_2 \times D_4) \rtimes C_2) \rtimes C_3$
(5,706)	14	[6,0,8]	$G(32, 6)$	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	(5,786)	5	[1,0,4]	$G(96, 189)$	$C_2 \times GL(2, 3)$
(5,707)	4	[2,0,2]	$G(32, 49)$	$(C_2 \times D_4) \rtimes C_2$	(5,787)	1	[1,0,0]	$G(12, 2)$	$C_{12}$
(5,708)	4	[1,0,3]	$G(32, 49)$	$(C_2 \times D_4) \rtimes C_2$	(5,788)	1	[1,0,0]	$G(12, 2)$	$C_{12}$
(5,709)	7	[3,0,4]	$G(32, 49)$	$(C_2 \times D_4) \rtimes C_2$	(5,789)	1	[1,0,0]	$G(24, 9)$	$C_{12} \times C_2$
(5,710)	4	[2,0,2]	$G(32, 7)$	$(C_8 \rtimes C_2) \rtimes C_2$	(5,790)	1	[1,0,0]	$G(24, 6)$	$D_{12}$
(5,711)	4	[2,0,2]	$G(32, 7)$	$(C_8 \rtimes C_2) \rtimes C_2$	(5,791)	1	[1,0,0]	$G(24, 6)$	$D_{12}$
(5,712)	4	[1,0,3]	$G(32, 7)$	$(C_8 \rtimes C_2) \rtimes C_2$	(5,792)	1	[1,0,0]	$G(24, 6)$	$D_{12}$
(5,713)	4	[2,0,2]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$	(5,793)	1	[1,0,0]	$G(48, 36)$	$C_2 \times D_{12}$
(5,714)	4	[1,0,3]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$	(5,794)	1	[1,0,0]	$G(12, 1)$	$C_3 \times C_4$
(5,715)	4	[2,0,2]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$	(5,795)	1	[1,0,0]	$G(12, 1)$	$C_3 \times C_4$
(5,716)	4	[1,0,3]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$	(5,796)	1	[1,0,0]	$G(144, 167)$	$C_6 \times ((C_6 \times C_2) \rtimes C_2)$
(5,717)	4	[1,0,3]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$	(5,797)	3	[2,0,1]	$G(144, 191)$	$C_3^2 \times (C_3^2 \rtimes C_4)$
(5,718)	4	[1,0,3]	$G(32, 43)$	$(C_2 \times D_4) \rtimes C_2$	(5,798)	2	[1,0,1]	$G(144, 192)$	$C_2^2 \times S_3^2$
(5,719)	4	[1,0,3]	$G(32, 48)$	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$	(5,799)	1	[1,0,0]	$G(144, 149)$	
(5,720)	4	[1,0,3]	$G(32, 37)$	$C_2 \times (C_8 \rtimes C_2)$	(5,800)	1	[1,0,0]	$G(144, 151)$	$C_2 \times ((C_6 \times S_3) \rtimes C_2)$

Table 16 (continued): birational classification of the algebraic  $k$ -tori of dimension 5

CARAT	#	$[s, r, u]$	$G(n, i)$	CARAT	#	$[s, r, u]$	$G(n, i)$		
(5,801)	6	[3,0,3]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,881)	1	[1,0,0]	$G(72, 21)$	$(C_3 \times (C_3 \times C_4)) \rtimes C_2$
(5,802)	2	[2,0,0]	$G(144, 115)$	$(C_2 \times (C_3^2 \rtimes C_4)) \rtimes C_2$	(5,882)	2	[1,0,1]	$G(72, 46)$	$C_2 \times S_3^2$
(5,803)	2	[2,0,0]	$G(144, 115)$	$(C_2 \times (C_3^2 \rtimes C_4)) \rtimes C_2$	(5,883)	1	[1,0,0]	$G(72, 23)$	$(C_6 \times S_3) \rtimes C_2$
(5,804)	2	[2,0,0]	$G(144, 115)$	$(C_2 \times (C_3^2 \rtimes C_4)) \rtimes C_2$	(5,884)	1	[1,0,0]	$G(72, 23)$	$(C_6 \times S_3) \rtimes C_2$
(5,805)	3	[2,0,1]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,885)	1	[1,0,0]	$G(72, 23)$	$(C_6 \times S_3) \rtimes C_2$
(5,806)	3	[2,0,1]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,886)	1	[1,0,0]	$G(72, 23)$	$(C_6 \times S_3) \rtimes C_2$
(5,807)	2	[2,0,0]	$G(144, 115)$	$(C_2 \times (C_3^2 \rtimes C_4)) \rtimes C_2$	(5,887)	5	[2,0,3]	$G(72, 46)$	$C_2 \times S_3^2$
(5,808)	3	[2,0,1]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,888)	4	[2,0,2]	$G(72, 45)$	$C_2 \times (C_3^2 \times C_4)$
(5,809)	3	[2,0,1]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,889)	6	[3,0,3]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,810)	4	[2,0,2]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,890)	6	[3,0,3]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,811)	4	[2,0,2]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,891)	6	[3,0,3]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,812)	4	[2,0,2]	$G(144, 186)$	$C_2 \times (S_3^2 \rtimes C_2)$	(5,892)	6	[3,0,3]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,813)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$	(5,893)	4	[2,0,2]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,814)	1	[1,0,0]	$G(144, 136)$	$(C_2 \times (C_3^2 \times C_4)) \rtimes C_2$	(5,894)	4	[2,0,2]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,815)	1	[1,0,0]	$G(144, 136)$	$(C_2 \times (C_3^2 \times C_4)) \rtimes C_2$	(5,895)	4	[2,0,2]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,816)	2	[2,0,0]	$G(144, 136)$	$(C_2 \times (C_3^2 \times C_4)) \rtimes C_2$	(5,896)	4	[2,0,2]	$G(72, 40)$	$S_{3,3}^2 \rtimes C_2$
(5,817)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$	(5,897)	1	[1,0,0]	$G(96, 209)$	$C_2 \times S_3 \times D_4$
(5,818)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$	(5,898)	1	[1,0,0]	$G(10, 2)$	$C_{10}$
(5,819)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$	(5,899)	1	[1,0,0]	$G(10, 2)$	$C_{10}$
(5,820)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$	(5,900)	3	[3,0,0]	$G(10, 1)$	$D_5$
(5,821)	1	[1,0,0]	$G(144, 154)$	$(C_2 \times S_3^2) \rtimes C_2$	(5,901)	3	[3,0,0]	$G(10, 1)$	$D_5$
(5,822)	5	[2,0,3]	$G(18, 3)$	$C_3 \times S_3$	(5,902)	2	[2,0,0]	$G(10, 2)$	$C_{10}$
(5,823)	5	[2,0,3]	$G(18, 3)$	$C_3 \times S_3$	(5,903)	1	[1,0,0]	$G(20, 5)$	$C_{10} \times C_2$
(5,824)	1	[1,0,0]	$G(24, 7)$	$C_2 \times (C_3 \rtimes C_4)$	(5,904)	3	[3,0,0]	$G(20, 4)$	$D_{10}$
(5,825)	1	[1,0,0]	$G(24, 10)$	$C_3 \times D_4$	(5,905)	1	[1,0,0]	$G(20, 4)$	$D_{10}$
(5,826)	2	[2,0,0]	$G(24, 10)$	$C_3 \times D_4$	(5,906)	2	[2,0,0]	$G(20, 4)$	$D_{10}$
(5,827)	1	[1,0,0]	$G(24, 10)$	$C_3 \times D_4$	(5,907)	1	[1,0,0]	$G(20, 4)$	$D_{10}$
(5,828)	2	[2,0,0]	$G(24, 8)$	$(C_6 \times C_2) \rtimes C_2$	(5,908)	1	[1,0,0]	$G(40, 13)$	$C_2^2 \times D_5$
(5,829)	2	[2,0,0]	$G(24, 8)$	$(C_6 \times C_2) \rtimes C_2$	(5,909)	2	[2,0,0]	$G(5, 1)$	$C_5$
(5,830)	2	[2,0,0]	$G(24, 8)$	$(C_6 \times C_2) \rtimes C_2$	(5,910)	4	[4,0,0]	$G(120, 35)$	$C_2 \times A_5$
(5,831)	2	[2,0,0]	$G(24, 8)$	$(C_6 \times C_2) \rtimes C_2$	(5,911)	4	[3,1,0]	$G(120, 34)$	$S_5$
(5,832)	1	[1,0,0]	$G(24, 5)$	$C_4 \times S_3$	(5,912)	4	[3,1,0]	$G(120, 34)$	$S_5$
(5,833)	1	[1,0,0]	$G(24, 5)$	$C_4 \times S_3$	(5,913)	2	[2,0,0]	$G(120, 35)$	$C_2 \times A_5$
(5,834)	1	[1,0,0]	$G(24, 5)$	$C_4 \times S_3$	(5,914)	2	[1,1,0]	$G(120, 34)$	$S_5$
(5,835)	1	[1,0,0]	$G(24, 5)$	$C_4 \times S_3$	(5,915)	2	[2,0,0]	$G(120, 35)$	$C_2 \times A_5$
(5,836)	3	[2,0,1]	$G(288, 1031)$	$C_2^2 \times (S_3^2 \rtimes C_2)$	(5,916)	2	[1,1,0]	$G(120, 34)$	$S_5$
(5,837)	2	[2,0,0]	$G(288, 880)$		(5,917)	6	[4,2,0]	$G(20, 3)$	$C_5 \times C_4$
(5,838)	2	[2,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$	(5,918)	6	[4,2,0]	$G(20, 3)$	$C_5 \times C_4$
(5,839)	1	[1,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$	(5,919)	4	[3,1,0]	$G(240, 189)$	$C_2 \times S_5$
(5,840)	1	[1,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$	(5,920)	2	[2,0,0]	$G(240, 190)$	$C_2^2 \times A_5$
(5,841)	2	[2,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$	(5,921)	2	[1,1,0]	$G(240, 189)$	$C_2 \times S_5$
(5,842)	1	[1,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$	(5,922)	2	[1,1,0]	$G(240, 189)$	$C_2 \times S_5$
(5,843)	1	[1,0,0]	$G(288, 889)$	$(C_2^2 \times S_3^2) \rtimes C_2$	(5,923)	2	[1,1,0]	$G(240, 189)$	$C_2 \times S_5$
(5,844)	1	[1,0,0]	$G(288, 941)$		(5,924)	2	[1,1,0]	$G(240, 189)$	$C_2 \times S_5$
(5,845)	1	[1,0,0]	$G(288, 977)$	$C_2 \times ((C_2 \times S_3^2) \rtimes C_2)$	(5,925)	2	[1,1,0]	$G(240, 189)$	$C_2 \times S_5$
(5,846)	5	[2,0,3]	$G(36, 12)$	$C_6 \times S_3$	(5,926)	6	[4,2,0]	$G(40, 12)$	$C_2 \times (C_5 \times C_4)$
(5,847)	2	[1,0,1]	$G(36, 12)$	$C_6 \times S_3$	(5,927)	2	[1,1,0]	$G(40, 12)$	$C_2 \times (C_5 \times C_4)$
(5,848)	2	[1,0,1]	$G(36, 12)$	$C_6 \times S_3$	(5,928)	4	[2,2,0]	$G(40, 12)$	$C_2 \times (C_5 \times C_4)$
(5,849)	1	[1,0,0]	$G(36, 6)$	$C_3 \times (C_3 \rtimes C_4)$	(5,929)	2	[1,1,0]	$G(40, 12)$	$C_2 \times (C_5 \times C_4)$
(5,850)	1	[1,0,0]	$G(36, 6)$	$C_3 \times (C_3 \rtimes C_4)$	(5,930)	2	[1,1,0]	$G(480, 1186)$	$C_2^2 \times S_5$
(5,851)	3	[1,0,2]	$G(36, 12)$	$C_6 \times S_3$	(5,931)	4	[4,0,0]	$G(60, 5)$	$A_5$
(5,852)	6	[3,0,3]	$G(36, 9)$	$C_3^2 \rtimes C_4$	(5,932)	2	[1,1,0]	$G(80, 50)$	$C_2^2 \times (C_5 \times C_4)$
(5,853)	6	[3,0,3]	$G(36, 9)$	$C_3^2 \rtimes C_4$	(5,933)	3	[1,0,2]	$G(160, 235)$	$C_2 \times (C_2^4 \times C_5)$
(5,854)	7	[3,0,4]	$G(36, 10)$	$S_3^2$	(5,934)	3	[1,0,2]	$G(160, 234)$	$(C_2^4 \times C_5) \rtimes C_2$
(5,855)	7	[3,0,4]	$G(36, 10)$	$S_3^2$	(5,935)	3	[1,0,2]	$G(160, 234)$	$(C_2^4 \times C_5) \rtimes C_2$
(5,856)	8	[3,0,5]	$G(36, 10)$	$S_3^2$	(5,936)	3	[1,0,2]	$G(1920, 240997)$	$C_2 \times (C_2^4 \times A_5)$
(5,857)	2	[2,0,0]	$G(48, 43)$	$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	(5,937)	3	[1,0,2]	$G(1920, 240996)$	$(C_2^4 \times A_5) \rtimes C_2$
(5,858)	1	[1,0,0]	$G(48, 45)$	$C_6 \times D_4$	(5,938)	3	[1,0,2]	$G(1920, 240996)$	$(C_2^4 \times A_5) \rtimes C_2$
(5,859)	1	[1,0,0]	$G(48, 35)$	$C_2 \times C_4 \times S_3$	(5,939)	3	[1,0,2]	$G(320, 1635)$	$(C_2^4 \times C_5) \rtimes C_4$
(5,860)	1	[1,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,940)	3	[1,0,2]	$G(320, 1635)$	$(C_2^4 \times C_5) \rtimes C_4$
(5,861)	2	[2,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,941)	3	[1,0,2]	$G(320, 1636)$	$C_2 \times ((C_2^4 \times C_5) \rtimes C_2)$
(5,862)	2	[2,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,942)	3	[1,0,2]	$G(3840, ?)$	$C_2 \times ((C_2^4 \times A_5) \rtimes C_2)$
(5,863)	1	[1,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,943)	3	[1,0,2]	$G(640, 21536)$	$C_2 \times ((C_2^4 \times C_5) \rtimes C_4)$
(5,864)	1	[1,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,944)	3	[1,0,2]	$G(80, 49)$	$C_2^4 \times C_5$
(5,865)	1	[1,0,0]	$G(48, 38)$	$D_4 \times S_3$	(5,945)	3	[1,0,2]	$G(960, 11358)$	$C_2^4 \times A_5$
(5,866)	1	[1,0,0]	$G(576, 8418)$	$C_2 \times ((C_2^2 \times S_3^2) \rtimes C_2)$	(5,946)	4	[2,0,2]	$G(120, 34)$	$S_5$
(5,867)	2	[1,0,1]	$G(72, 48)$	$C_2 \times C_6 \times S_3$	(5,947)	4	[2,0,2]	$G(120, 34)$	$S_5$
(5,868)	1	[1,0,0]	$G(72, 29)$	$C_6 \times (C_3 \rtimes C_4)$	(5,948)	4	[0,0,4]	$G(120, 35)$	$C_2 \times A_5$
(5,869)	7	[3,0,4]	$G(72, 46)$	$C_2 \times S_3^2$	(5,949)	4	[0,0,4]	$G(1440, 5842)$	$C_2 \times S_6$
(5,870)	6	[3,0,3]	$G(72, 45)$	$C_2 \times (C_3^2 \rtimes C_4)$	(5,950)	4	[0,0,4]	$G(240, 189)$	$C_2 \times S_5$
(5,871)	1	[1,0,0]	$G(72, 30)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	(5,951)	4	[1,0,3]	$G(360, 118)$	$A_6$
(5,872)	1	[1,0,0]	$G(72, 30)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	(5,952)	4	[2,0,2]	$G(60, 5)$	$A_5$
(5,873)	1	[1,0,0]	$G(72, 30)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	(5,953)	4	[1,0,3]	$G(720, 763)$	$S_6$
(5,874)	1	[1,0,0]	$G(72, 30)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	(5,954)	4	[0,0,4]	$G(720, 763)$	$S_6$
(5,875)	1	[1,0,0]	$G(72, 21)$	$(C_3 \times (C_3 \times C_4)) \rtimes C_2$	(5,955)	4	[0,0,4]	$G(720, 766)$	$C_2 \times A_6$
(5,876)	3	[2,0,1]	$G(72, 45)$	$C_2 \times (C_3^2 \rtimes C_4)$					
(5,877)	3	[2,0,1]	$G(72, 45)$	$C_2 \times (C_3^2 \rtimes C_4)$					
(5,878)	2	[1,0,1]	$G(72, 46)$	$C_2 \times S_3^2$					
(5,879)	2	[1,0,1]	$G(72, 46)$	$C_2 \times S_3^2$					
(5,880)	1	[1,0,0]	$G(72, 21)$	$(C_3 \times (C_3 \times C_4)) \rtimes C_2$					

## REFERENCES

- [Arn84] J. E. Arnold, Jr., *Groups of permutation projective dimension two*, Proc. Amer. Math. Soc. **91** (1984) 505–509.
- [Azu50] G. Azumaya, *Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem*, Nagoya Math. J. **1** (1950) 117–124.
- [Ben91] D. J. Benson, *Representations and cohomology. I*, Basic representation theory of finite groups and associative algebras. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, Cambridge, 1991.
- [Bog88] F. A. Bogomolov, *The Brauer group of quotient spaces of linear representations*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987) 485–516, 688; translation in Math. USSR-Izv. **30** (1988) 455–485.
- [Bog90] F. A. Bogomolov, *Brauer groups of the fields of invariants of algebraic groups*, (Russian) Mat. Sb. **180** (1989) 279–293; translation in Math. USSR-Sb. **66** (1990) 285–299.
- [BMP04] F. A. Bogomolov, J. Maciel, T. Petrov, *Unramified Brauer groups of finite simple groups of Lie type  $A_l$* , Amer. J. Math. **126** (2004) 935–949.
- [Bou00] S. Bouc, *Burnside rings*, Handbook of algebra, vol. 2, 739–804, North-Holland, Amsterdam, 2000.
- [BBNWZ78] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, H. Zassenhaus. *Crystallographic Groups of Four-Dimensional Space*, John Wiley, New York, 1978.
- [Bro82] K. S. Brown, *Cohomology of Groups*, Grad. Texts in Math., vol. 87, Springer-Verlag, 1972.
- [BM83] G. Butler, J. McKay, *The transitive groups of degree up to eleven*, Comm. Algebra **11** (1983) 863–911.
- [Carat] J. Opgenorth, W. Plesken, T. Schulz, CARAT, GAP 4 package, version 2.1b1, 2008, available at <http://wwwb.math.rwth-aachen.de/carat/>.
- [CE56] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.
- [CHKK10] H. Chu, S. Hu, M. Kang, B. E. Kunyavskii, *Noether's problem and the unramified Brauer group for groups of order 64*, Int. Math. Res. Not. IMRN **2010** 2329–2366.
- [CHKP08] H. Chu, S. Hu, M. Kang, Y. G. Prokhorov, *Noether's problem for groups of order 32*, J. Algebra **320** (2008) 3022–3035.
- [CK01] H. Chu, M. Kang, *Rationality of  $p$ -group actions*, J. Algebra **237** (2001) 673–690.
- [CK00] A. Cortella, B. Kunyavskii, *Rationality problem for generic tori in simple groups*, J. Algebra **225** (2000) 771–793.
- [CTS77] J.-L. Colliot-Thélène, J.-J. Sansuc, *La  $R$ -équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977) 175–229.
- [CTS87] J.-L. Colliot-Thélène, J.-J. Sansuc, *Principal homogeneous spaces under flasque tori: Applications*, J. Algebra **106** (1987) 148–205.
- [CR81] C. W. Curtis, I. Reiner, *Methods of representation theory, vol. I, with applications to finite groups and orders*, Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1981.
- [CR87] C. W. Curtis, I. Reiner, *Methods of representation theory, vol. II, With applications to finite groups and orders*, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987.
- [Dre70] A. Dress, *On the Krull-Schmidt theorem for integral group representations of rank 1*, Michigan Math. J. **17** (1970) 273–277.
- [Dre73] A. W. M. Dress, *Contributions to the theory of induced representations*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp.183–240. Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973.
- [Dre75] A. W. M. Dress, *The permutation class group of a finite group*, J. Pure Appl. Algebra **6** (1975) 1–12.
- [End11] S. Endo, *The rationality problem for norm one tori*, Nagoya Math. J. **202** (2011) 83–106.
- [End12] S. Endo, private communications, 2012.
- [EH79] S. Endo, Y. Hironaka, *Finite groups with trivial class groups*, J. Math. Soc. Japan **31** (1979) 161–174.
- [EM73] S. Endo, T. Miyata, *Invariants of finite abelian groups*, J. Math. Soc. Japan **25** (1973) 7–26.
- [EM74] S. Endo, T. Miyata, *On a classification of the function fields of algebraic tori*, Nagoya Math. J. **56** (1975) 85–104.
- [EM82] S. Endo, T. Miyata, *Integral representations with trivial first cohomology groups*, Nagoya Math. J. **85** (1982) 231–240.
- [Fac03] A. Facchini, *The Krull-Schmidt theorem*, Handbook of algebra, Vol. 3, 357–397, North-Holland, Amsterdam, 2003.
- [Flo] M. Florence, *Non rationality of some norm-one tori*, preprint (2006).
- [GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (<http://www.gap-system.org>).
- [GW93] R. M. Guralnick, A. Weiss, *Transitive permutation lattices in the same genus and embeddings of groups*, Linear algebraic groups and their representations (Los Angeles, CA, 1992), 21–33, Contemp. Math., 153, Amer. Math. Soc., Providence, RI, 1993.
- [Haj83] M. Hajja, *A note on monomial automorphisms*, J. Algebra **85** (1983) 243–250.
- [Haj87] M. Hajja, *Rationality of finite groups of monomial automorphisms of  $k(x, y)$* , J. Algebra **109** (1987) 46–51.
- [HK92] M. Hajja, M. Kang, *Finite group actions on rational function fields*, J. Algebra **149** (1992) 139–154.
- [HK94] M. Hajja, M. Kang, *Three-dimensional purely monomial group actions*, J. Algebra **170** (1994) 805–860.
- [HAP] G. Ellis, HAP, GAP 4 package, version 1.10.6, 2012, available at <http://hamilton.nuigalway.ie/Hap/www/>.
- [Hür84] W. Hürlimann, *On algebraic tori of norm type*, Comment. Math. Helv. **59** (1984) 539–549.
- [HKO98] P. Hindman, L. Klingler, C. J. Odenthal, *On the Krull-Schmidt-Azumaya theorem for integral group rings*, Comm. Algebra **26** (1998) 3743–3758.
- [HK10] A. Hoshi, M. Kang, *Twisted symmetric group actions*, Pacific J. Math. **248** (2010) 285–304.
- [HKKi] A. Hoshi, M. Kang, H. Kitayama, *Quasi-monomial actions and some 4-dimensional rationality problems*, arXiv:1201.1332.
- [HKKu] A. Hoshi, M. Kang, B. E. Kunyavskii, *Noether's problem and unramified Brauer groups*, to appear in Asian J. Math., arXiv:1202.5812.
- [HKY11] A. Hoshi, H. Kitayama, A. Yamasaki, *Rationality problem of three-dimensional monomial group actions*, J. Algebra **341** (2011) 45–108.

- [HR08] A. Hoshi, Y. Rikuna, *Rationality problem of three-dimensional purely monomial group actions: the last case*, Math. Comp. **77** (2008) 1823–1829.
- [Jon65] A. Jones, *On representations of finite groups over valuation rings*, Illinois J. Math. **9** (1965) 297–303.
- [Kan09] M. Kang, *Retract rationality and Noether’s problem*, Int. Math. Res. Not. IMRN **2009** 2760–2788.
- [Kan12] M. Kang, *Retract rational fields*, J. Algebra **349** (2012) 22–37.
- [KP10] M. Kang, Y. G. Prokhorov, *Rationality of three-dimensional quotients by monomial actions*, J. Algebra **324** (2010) 2166–2197.
- [Knu98] D. E. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, Addison-Wesley, third edition, 1998.
- [Kun90] B. E. Kunyavskii, *Three-dimensional algebraic tori*, Selecta Math. Soviet. **9** (1990) 1–21.
- [Kun07] B. E. Kunyavskii, *Algebraic tori — thirty years after*, Vestnik Samara State Univ. (2007) 198–214.
- [Kun10] B. E. Kunyavskii, *The Bogomolov multiplier of finite simple groups*, Cohomological and geometric approaches to rationality problems, 209–217, Progr. Math., 282, Birkhäuser Boston, Inc., Boston, MA, 2010.
- [LL00] N. Lemire, M. Lorenz, *On certain lattices associated with generic division algebras*, J. Group Theory **3** (2000) 385–405.
- [LeB95] L. Le Bruyn, *Generic norm one tori*, Nieuw Arch. Wisk. (4) **13** (1995) 401–407.
- [Len74] H. W. Lenstra, Jr., *Rational functions invariant under a finite abelian group*, Invent. Math. **25** (1974) 299–325.
- [Lor05] M. Lorenz, *Multiplicative invariant theory*, Encyclopaedia Math. Sci., vol. 135, Springer-Verlag, Berlin, 2005.
- [MN98] M. Matsumoto, T. Nishimura, *Mersenne Twister: A 623-dimensionally equidistributed uniform pseudorandom number generator*, ACM Trans. on Modeling and Computer Simulation **8** (1998) 3–30.
- [Mo] P. Moravec, *Unramified Brauer groups of finite and infinite groups*, to appear in Amer. J. Math.
- [Ple78] W. Plesken, *On reducible and decomposable representations of orders*, J. Reine Angew. Math. **297** (1978) 188–210.
- [Ple85] W. Plesken, *Finite unimodular groups of prime degree and circulants*, J. Algebra **97** (1985) 286–312.
- [PH84] W. Plesken, W. Hanrath, *The lattices of six-dimensional Euclidean space*, Math. Comp. **43** (1984) 573–587.
- [PP77] W. Plesken, M. Pohst, *On maximal finite irreducible subgroups of  $GL(n, \mathbf{Z})$ . I. The five and seven dimensional cases. II. The six dimensional case*, Math. Comp. **31** (1977) 536–551, 552–573.
- [PP80] W. Plesken and M. Pohst, *On maximal finite irreducible subgroups of  $GL(n, \mathbf{Z})$ . III. The nine dimensional case. IV. Remarks on even dimensions with application to  $n = 8$ . V. The eight dimensional case and a complete description of dimensions less than ten*, Math. Comp. **34** (1980) 245–258, 259–275, 277–301.
- [PS00] W. Plesken, T. Schulz, *Counting crystallographic groups in low dimensions*, Exp. Math. **9** (2000) 407–411.
- [Pop98] S. Yu. Popov, *Galois lattices and their birational invariants*. (Russian) Vestn. Samar. Gos. Univ. Mat. Mekh. Fiz. Khim. Biol. 1998, no. 4, 71–83.
- [Sal84a] D. J. Saltman, *Retract rational fields and cyclic Galois extensions*, Israel J. Math. **47** (1984) 165–215.
- [Sal84b] D. J. Saltman, *Noether’s problem over an algebraically closed field*, Invent. Math. **77** (1984) 71–84.
- [Sal87] D. J. Saltman, *Multiplicative field invariants*, J. Algebra **106** (1987) 221–238.
- [Sal90] D. J. Saltman, *Multiplicative field invariants and the Brauer group*, J. Algebra **133** (1990) 533–544.
- [Sou94] B. Souvignier, *Irreducible finite integral matrix groups of degree 8 and 10*, Math. Comp. **63** (1994) 335–350.
- [Swa60] R. G. Swan, *Induced Representations and Projective Modules*, Ann. Math. **71** (1960) 552–578.
- [Swa83] R. G. Swan, *Noether’s problem in Galois theory*, Emmy Noether in Bryn Mawr (Bryn Mawr, Pa., 1982), 21–40, Springer, New York-Berlin, 1983.
- [Swa88] R. G. Swan, *Torsion free cancellation over orders*, Illinois J. Math. **32** (1988) 329–360.
- [Swa10] R. G. Swan, *The flabby class group of a finite cyclic group*, Fourth International Congress of Chinese Mathematicians, 259–269, AMS/IP Stud. Adv. Math., 48, Amer. Math. Soc., Providence, RI, 2010.
- [Vos67] V. E. Voskresenskii, *On two-dimensional algebraic tori II*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967) 711–716; translation in Math. USSR-Izv. **1** (1967) 691–696.
- [Vos70] V. E. Voskresenskii, *Birational properties of linear algebraic groups*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970) 3–19; translation in Math. USSR-Izv. **4** (1970) 1–17.
- [Vos74] V. E. Voskresenskii, *Stable equivalence of algebraic tori*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 3–10; translation in Math. USSR-Izv. **8** (1974) 1–7.
- [Vos98] V. E. Voskresenskii, *Algebraic groups and their birational invariants*, Translated from the Russian manuscript by Boris Kunyavskii, Translations of Mathematical Monographs, 179, American Mathematical Society, Providence, RI, 1998.
- [Yam12] A. Yamasaki, *Negative solutions to three-dimensional monomial Noether problem*, J. Algebra **370** (2012) 46–78.

DEPARTMENT OF MATHEMATICS, RIKKYO UNIVERSITY, JAPAN

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, JAPAN